Chapter 1 Exercises

What is a Vector? From the Real Line to $\mathbb{R}^n$

1 Vectors in $\mathbb{R}^n$: Notation and Equality

Exercise 1. Find x, y, and z so that $(x+1,\ 2y,\ 3) = (4,\ 6,\ z)$.

Exercise 2. Write down the zero vector of $\mathbb{R}^5$. Then explain in one sentence why $\mathbf{0} \in \mathbb{R}^2$ and $\mathbf{0} \in \mathbb{R}^3$ are *not* the same vector, even though both are "all zeros."

Exercise 3. Find all values of t so that $(t^2,\ 5) = (4,\ 5)$ in $\mathbb{R}^2$.

Exercise 4. Two vectors in $\mathbb{R}^5$ agree in every component except the third: one has 7 in that spot, the other has -7. Could these two vectors ever be equal, no matter what the other components are? Explain.

2 Scalar Multiplication

Exercise 1. Let $\mathbf{u} = (2, -3, 5)$ and $\mathbf{v} = (-1, 0, 2, 4)$. Compute:

(a) $4\mathbf{u}$

(b) $-3\mathbf{v}$

(c) $\dfrac{1}{2}\mathbf{u}$

(d) $0 \cdot \mathbf{v}$

Exercise 2. Let $\mathbf{v} = (4, -6)$. Find the scalar c such that $c\mathbf{v} = (-6, 9)$.

Exercise 3. Let $\mathbf{v} = (3, -4)$. For each value of c, state whether $c\mathbf{v}$ points in the **same** direction as $\mathbf{v}$, the **opposite** direction, or is the **zero vector**.

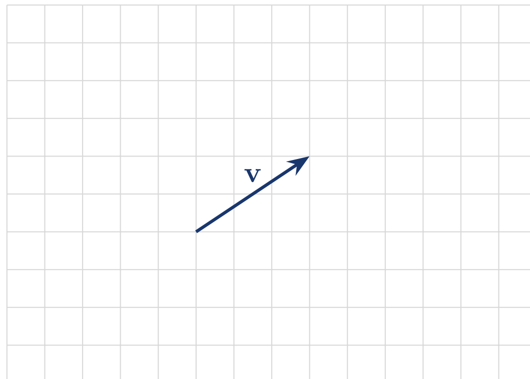
(a) $c = 5$

(b) $c = -2$

(c) $c = 0$

(d) $c = \frac{1}{3}$

Exercise 4. The vector $\mathbf{v}$ is drawn on the grid below. On your own copy of this grid, sketch $2\mathbf{v}$, $-\mathbf{v}$, $\frac{1}{2}\mathbf{v}$, and $-2\mathbf{v}$, each starting from a different point so they don't overlap. Label each one.



Exercise 5. Suppose $c\mathbf{u} = \mathbf{0}$ and $\mathbf{u} \neq \mathbf{0}$. What must c be? Justify your answer using components (i.e. explain why no other value of c could work).

3 Addition and Subtraction

Exercise 1. Compute $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ for each pair.

(a) $\mathbf{u} = (3, -2)$, $\mathbf{v} = (-1, 5)$

(b) $\mathbf{u} = (2, 0, -3)$, $\mathbf{v} = (-4, 1, 1)$

(c) $\mathbf{u} = (1, -1, 2, 3)$, $\mathbf{v} = (0, 4, -2, 1)$

Exercise 2. If $\mathbf{u} + \mathbf{v} = (5, 1)$ and $\mathbf{u} = (3, -2)$, find $\mathbf{v}$.

Exercise 3. Sketch, on your own grid, the tip-to-tail construction of $\mathbf{u} + \mathbf{v}$ for $\mathbf{u} = (3, 1)$ and $\mathbf{v} = (1, 2)$. Then sketch the parallelogram construction for the same two vectors, and check that both give the same sum.

Exercise 4. Using only the properties of addition (commutativity, associativity, the zero vector, additive inverses — *not* components), simplify:

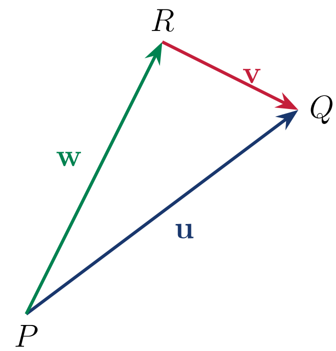
$$(\mathbf{u} + \mathbf{v}) - \mathbf{v} \qquad \text{and} \qquad \mathbf{u} + (-\mathbf{u}) + \mathbf{w}.$$

Exercise 5. Solve for the vector $\mathbf{x} \in \mathbb{R}^3$:

$$2\mathbf{x} + (1, -2, 3) = (7, 0, -1).$$

Exercise 6. Let $\mathbf{u} = (2, -1)$, $\mathbf{v} = (-3, 4)$, $\mathbf{w} = (1, 1)$. Compute $\mathbf{u} + 2\mathbf{v} - 3\mathbf{w}$.

Exercise 7. In the diagram below, $\mathbf{u}$ and $\mathbf{w}$ both start at P, and $\mathbf{v}$ goes from R to Q. Without using any coordinates, express $\mathbf{v}$ in terms of $\mathbf{u}$ and $\mathbf{w}$.



Hint

Going from P to Q, you can travel directly (that's $\mathbf{u}$), or via R (that's $\mathbf{w}$ then $\mathbf{v}$). Chasles relates these two routes.

4 From Two Points to a Vector

Exercise 1. Compute $\overrightarrow{AB}$ for each pair of points.

(a) $A = (2, -1)$, $B = (5, 3)$

(b) $A = (1, 0, -2)$, $B = (-3, 4, 1)$

Exercise 2. Let $A = (-2, 3)$. Find the point B such that $\overrightarrow{AB} = (4, -1)$.

Exercise 3. Let $B = (6, 2)$. Find the point A such that $\overrightarrow{AB} = (-3, 5)$.

Be careful

Be careful with the direction of subtraction: $\overrightarrow{AB} = B - A$, not $A - B$.

Exercise 4. Let $A = (3, -2, 1)$ and $B = (5, 0, -3)$ in $\mathbb{R}^3$. Compute $\overrightarrow{AB}$ and $\overrightarrow{BA}$, and verify that $\overrightarrow{BA} = -\overrightarrow{AB}$.

Exercise 5. Let $A = (1, 1)$ and $B = (4, 2)$. Find the point M such that $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB}$ (i.e. M is reached by going *halfway* from A to B).

5 The Chasles Relation

Exercise 1. Let $A = (1, 2)$, $B = (4, -1)$, $C = (6, 3)$. Compute $\overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{AC}$ separately, and verify that $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

Exercise 2. Suppose $\overrightarrow{PQ} = (2, -5)$ and $\overrightarrow{QR} = (-1, 3)$. Without knowing the coordinates of P, Q, R, find $\overrightarrow{PR}$.

Exercise 3. Starting only from the Chasles relation $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$, derive the identity $\overrightarrow{BA} = -\overrightarrow{AB}$.

Hint

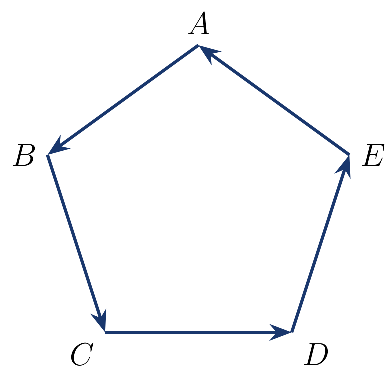
Set $C = A$ in the Chasles relation, and use $\overrightarrow{AA} = \mathbf{0}$.

Exercise 4. Let $A = (0, 0)$, $B = (2, 1)$, $C = (5, 2)$, $D = (6, 5)$. Compute $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, and $\overrightarrow{AD}$ separately, and verify that

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD}.$$

Exercise 5. Suppose $\overrightarrow{AB} = \overrightarrow{CD}$. Using only identities from this chapter (not coordinates), show that $\overrightarrow{BA} = \overrightarrow{DC}$.

Exercise 6. The diagram below shows a closed path that visits A, B, C, D, E in order and returns to A.

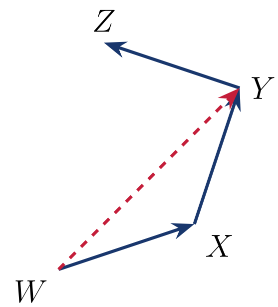


What is $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EA}$? Justify your answer using the Chasles relation — don't just guess from the picture.

Hint

Combine the vectors two at a time, left to right: first $\overrightarrow{AB} + \overrightarrow{BC}$, then add $\overrightarrow{CD}$ to that result, and so on.

Exercise 7. In the diagram below, the dashed vector $\overrightarrow{WY}$ is a "shortcut" across the path $W \to X \to Y$.



(a) Express $\overrightarrow{WZ}$ in terms of $\overrightarrow{WY}$ and $\overrightarrow{YZ}$ (the "short" route).

(b) Express $\overrightarrow{WZ}$ in terms of $\overrightarrow{WX}$, $\overrightarrow{XY}$, and $\overrightarrow{YZ}$ (the "long" route).

(c) Now suppose you're told $\overrightarrow{WX} = (3, 1)$, $\overrightarrow{XY} = (1, 3)$, and $\overrightarrow{WZ} = (1, 5)$, but *not* $\overrightarrow{YZ}$. Use your answer to (b) to find $\overrightarrow{YZ}$ algebraically — don't read it off the picture.

(d) **(Challenge)** Parts (a) and (b) both describe $\overrightarrow{WZ}$, so they must be equal: $\overrightarrow{WY} + \overrightarrow{YZ} = \overrightarrow{WX} + \overrightarrow{XY} + \overrightarrow{YZ}$. This diagram uses specific points, but the fact itself is completely general. Let W, X, Y, Z be *any* four points in $\mathbb{R}^n$ (position vectors, not the ones pictured). Prove algebraically, using only $\overrightarrow{PQ} = Q - P$, that

$$\overrightarrow{WX} + \overrightarrow{XY} + \overrightarrow{YZ} = \overrightarrow{WY} + \overrightarrow{YZ}$$

always holds — i.e. that the "short" and "long" routes always agree, no matter where the four points are.

Hint

For (c), rearrange the equation from (b) to isolate $\overrightarrow{YZ}$. For (d), expand every $\overrightarrow{PQ}$ as $Q - P$ on both sides and watch everything cancel.

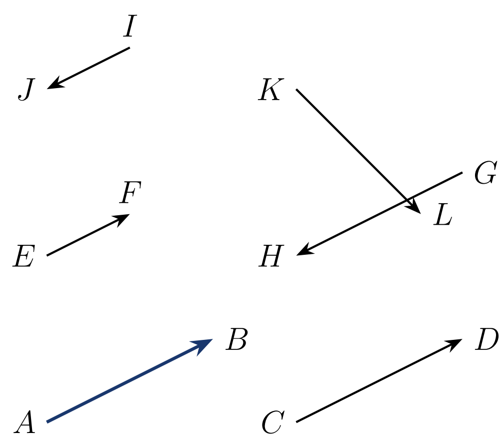
6 Equal, Collinear, and Parallel Vectors

Exercise 1. In the diagram below, each arrow is a vector. Using only what the picture shows (no coordinates are given),

(a) Which vectors are **parallel** to $\overrightarrow{AB}$?

(b) Which vectors are **equal (equivalent)** to $\overrightarrow{AB}$?

(c) Which vector is **opposite** to $\overrightarrow{AB}$?



Exercise 2. For each pair, state whether $\mathbf{u}$ and $\mathbf{v}$ are **equal**, **collinear but not equal**, or **neither**.

(a) $\mathbf{u} = (2, -3)$, $\mathbf{v} = (-6, 9)$

(b) $\mathbf{u} = (1, 4)$, $\mathbf{v} = (2, 7)$

(c) $\mathbf{u} = (2, -3)$, $\mathbf{v} = (2, -3)$

(d) $\mathbf{u} = (3, 1, 2)$, $\mathbf{v} = (-6, -2, -4)$

Exercise 3. Find all values of k for which $\mathbf{u} = (k, -3)$ and $\mathbf{v} = (4, 6)$ are collinear.

Exercise 4. Three points $A = (1, 1)$, $B = (3, 5)$, $C = (6, 11)$ are given. A quick way to test whether three points lie on a single straight line is to check whether $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are collinear. Carry out this test: are A, B, C collinear?

Hint

A, B, C are collinear exactly when $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are collinear vectors (both point along the same line through A).

Exercise 5. Repeat the test from the previous exercise for $A = (0, 0)$, $B = (2, 3)$, $C = (5, 6)$. Are these three points collinear?

7 Parallelograms and Parallelepipeds

Exercise 1. Let $A = (0, 0)$, $B = (4, 1)$, $C = (6, 5)$ be three vertices of a parallelogram $ABCD$ (in that cyclic order). Find the coordinates of D.

Hint

In a parallelogram $ABCD$, opposite sides are equal as vectors: $\overrightarrow{AD} = \overrightarrow{BC}$.

Exercise 2. Let $A = (1, 1)$, $B = (5, 2)$, $C = (6, 6)$, $D = (2, 5)$. Verify that $ABCD$ is a parallelogram by checking $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$.

Exercise 3. A parallelepiped has vertex $A = (1, 1, 0)$ and edge vectors $\mathbf{u} = (2, 0, 0)$, $\mathbf{v} = (0, 3, 0)$, $\mathbf{w} = (0, 0, 4)$ from A. Find the coordinates of the opposite vertex G (the far corner reached by following all three edges).

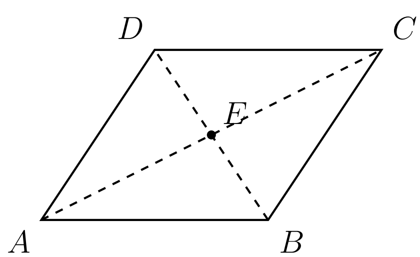
Exercise 4. Three vertices of a parallelogram $ABCD$ (in cyclic order) are $A = (2, -1)$, $B = (5, 1)$, $D = (0, 3)$. Find the coordinates of C.

Hint

Here A, B, D are known and C is missing — use $\overrightarrow{AB} = \overrightarrow{DC}$ instead.

Exercise 5. A parallelepiped has vertex $A = (0, 0, 0)$ and edge vectors $\mathbf{u} = (1, 2, 0)$, $\mathbf{v} = (0, 1, 3)$, $\mathbf{w} = (2, 0, 1)$ from A. Find the coordinates of the three vertices adjacent to A (namely $A + \mathbf{u}$, $A + \mathbf{v}$, $A + \mathbf{w}$) and the opposite vertex G.

Exercise 6. $ABCD$ is a parallelogram (cyclic order), and E is the point where diagonals AC and BD cross.



(a) Write A, B, C, D for the position vectors of the vertices, and let $E = \frac{1}{2}(A+C)$ (the midpoint of diagonal AC). Using the parallelogram property $\overrightarrow{AB} = \overrightarrow{DC}$, show algebraically that $E = \frac{1}{2}(B + D)$ as well — i.e. E is *also* the midpoint of diagonal BD. This is why a single point E can be marked at the crossing of both diagonals.

(b) Now use part (a), the Chasles relation, and the identity $\overrightarrow{AD} = \overrightarrow{BC}$ to reduce each expression to a single named vector.

(i) $\overrightarrow{AE} + \overrightarrow{EB}$

(ii) $\overrightarrow{BC} + \overrightarrow{BA}$

(iii) $\overrightarrow{AE} + \overrightarrow{AE}$

(iv) $\overrightarrow{AB} - \overrightarrow{DB}$

Hint

For (b)(ii), rewrite $\overrightarrow{BA}$ using $\overrightarrow{AD} = \overrightarrow{BC}$ and Chasles through A. For (b)(iv), remember $-\overrightarrow{DB} = \overrightarrow{BD}$.

8 Synthesis and Challenge Problems

Exercise 1. Let $\mathbf{u} = (1, -2, 3)$ and $\mathbf{v} = (0, 4, -1)$. Compute $2\mathbf{u} - 3\mathbf{v}$, showing every step.

Exercise 2. Points A, B, C satisfy $\overrightarrow{AB} = 2\mathbf{w}$ and $\overrightarrow{AC} = 5\mathbf{w}$ for some nonzero vector $\mathbf{w}$. Explain why A, B, C must lie on a single straight line.

Exercise 3. Explain, in your own words, why it does not make sense to add a vector in $\mathbb{R}^2$ to a vector in $\mathbb{R}^3$. What property of the definition of vector addition breaks down?

Exercise 4. (Challenge) Let A, B, C, D be four points with $\overrightarrow{AB} = \overrightarrow{DC}$. Show algebraically that this forces $\overrightarrow{AD} = \overrightarrow{BC}$ as well (i.e. that $ABCD$ is automatically a parallelogram).

Hint

Write everything in terms of the position vectors of A, B, C, D (their coordinates), and use $\overrightarrow{XY} = Y - X$.

Exercise 5. Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ satisfy $\mathbf{u} + \mathbf{v} = \mathbf{0}$. Show directly from the component-wise definition of addition (not by quoting the "additive inverse" property) that $\mathbf{v} = -\mathbf{u}$.

Exercise 6. Let $\mathbf{u} = (2, k)$ and $\mathbf{v} = (k, 8)$. Find *all* values of k for which $\mathbf{u}$ and $\mathbf{v}$ are collinear.

Hint

Collinear means $\mathbf{v} = \lambda\mathbf{u}$ for some scalar λ. Write both equations this gives you, then eliminate λ.