Chapter 1 Exercises - Solutions

What is a Vector? From the Real Line to $\mathbb{R}^n$

Solutions

Section 1: Vectors in $\mathbb{R}^n$

Solution

Matching components: $x + 1 = 4 \Rightarrow x = 3$; $2y = 6 \Rightarrow y = 3$; $z = 3$.

Solution

$\mathbf{0} \in \mathbb{R}^5 = (0, 0, 0, 0, 0)$. The zero vector of $\mathbb{R}^2$ has 2 components and the zero vector of $\mathbb{R}^3$ has 3 components — they live in different spaces (ordered tuples with a different number of components), so they cannot be compared or be equal, even though every entry is 0 in both.

Solution

$t^2 = 4 \Rightarrow t = 2$ or $t = -2$. Both work: $(2^2, 5) = (4, 5)$ and $((-2)^2, 5) = (4, 5)$.

Solution

No. Equality requires *every* component to match, and the third components are 7 and -7: since $7 \neq -7$, the vectors can never be equal, no matter what the other four components are.

Section 2: Scalar Multiplication

Solution

(a) $4\mathbf{u} = (8, -12, 20)$

(b) $-3\mathbf{v} = (3, 0, -6, -12)$

(c) $\frac{1}{2}\mathbf{u} = (1,\ -1.5,\ 2.5)$

(d) $0 \cdot \mathbf{v} = (0, 0, 0, 0) = \mathbf{0}$

Solution

We need $c(4, -6) = (-6, 9)$, so $4c = -6 \Rightarrow c = -\frac{3}{2}$. Check: $-\frac{3}{2}(-6) = 9$. ✓

Solution

(a) $c = 5 > 0$: same direction ($5\mathbf{v} = (15, -20)$).

(b) $c = -2 < 0$: opposite direction ($-2\mathbf{v} = (-6, 8)$).

(c) $c = 0$: zero vector ($0\mathbf{v} = (0, 0)$).

(d) $c = \frac{1}{3} > 0$: same direction ($\frac{1}{3}\mathbf{v} = (1, -\frac{4}{3})$).

Solution

Taking $\mathbf{v}$ to run from $(5, 4)$ to $(8, 6)$: $2\mathbf{v}$ is twice as long, same direction (e.g. from $(0, 0)$ to $(6, 4)$); $-\mathbf{v}$ is the same size, reversed (e.g. from $(8, 6)$ to $(5, 4)$); $\frac{1}{2}\mathbf{v}$ is half as long, same direction; $-2\mathbf{v}$ is twice as long, reversed. In every case the arrow's size scales by $|c|$ and its *sense* flips exactly when $c < 0$ — where you choose to draw the tail doesn't matter, only the size and direction of the arrow.

Solution

c must be 0. Writing $\mathbf{u} = (u_1, \ldots, u_n)$ with some $u_i \neq 0$ (since $\mathbf{u} \neq \mathbf{0}$), the i-th component of $c\mathbf{u}$ is cu_i. For this to equal the i-th component of $\mathbf{0}$, which is 0, we need $cu_i = 0$; since $u_i \neq 0$, this forces $c = 0$.

Section 3: Addition and Subtraction

Solution

(a) $\mathbf{u} + \mathbf{v} = (2, 3)$, $\quad \mathbf{u} - \mathbf{v} = (4, -7)$

(b) $\mathbf{u} + \mathbf{v} = (-2, 1, -2)$, $\quad \mathbf{u} - \mathbf{v} = (6, -1, -4)$

(c) $\mathbf{u} + \mathbf{v} = (1, 3, 0, 4)$, $\quad \mathbf{u} - \mathbf{v} = (1, -5, 4, 2)$

Solution

$\mathbf{v} = (\mathbf{u} + \mathbf{v}) - \mathbf{u} = (5, 1) - (3, -2) = (2, 3)$.

Solution

Both constructions — tip-to-tail and the parallelogram rule — give the diagonal $\mathbf{u} + \mathbf{v} = (4, 3)$. If your two sketches land on different endpoints, re-check that $\mathbf{v}$ is drawn starting from the *tip* of $\mathbf{u}$ (tip-to-tail), not from the origin.

Solution

$(\mathbf{u} + \mathbf{v}) - \mathbf{v} = \mathbf{u} + (\mathbf{v} - \mathbf{v}) = \mathbf{u} + \mathbf{0} = \mathbf{u}$.
$\mathbf{u} + (-\mathbf{u}) + \mathbf{w} = \mathbf{0} + \mathbf{w} = \mathbf{w}$.

Solution

$2\mathbf{x} = (7, 0, -1) - (1, -2, 3) = (6, 2, -4)$, so $\mathbf{x} = (3, 1, -2)$.

Solution

$\mathbf{u} + 2\mathbf{v} - 3\mathbf{w} = (2, -1) + (-6, 8) - (3, 3) = (2 - 6 - 3,\ -1 + 8 - 3) = (-7, 4)$.

Solution

Going from P to Q directly gives $\mathbf{u}$. Going from P to Q via R gives $\mathbf{w}$ followed by $\mathbf{v}$, which by Chasles is $\mathbf{w} + \mathbf{v}$. Both routes reach the same point, so $\mathbf{w} + \mathbf{v} = \mathbf{u}$, which gives

$$\mathbf{v} = \mathbf{u} - \mathbf{w}.$$

Section 4: From Two Points to a Vector

Solution

(a) $\overrightarrow{AB} = B - A = (5 - 2,\ 3 - (-1)) = (3, 4)$.

(b) $\overrightarrow{AB} = B - A = (-3 - 1,\ 4 - 0,\ 1 - (-2)) = (-4, 4, 3)$.

Solution

$B = A + \overrightarrow{AB} = (-2, 3) + (4, -1) = (2, 2)$.

Solution

$A = B - \overrightarrow{AB} = (6, 2) - (-3, 5) = (9, -3)$.

Solution

$\overrightarrow{AB} = B - A = (2, 2, -4)$. $\overrightarrow{BA} = A - B = (-2, -2, 4)$. Indeed $\overrightarrow{BA} = -\overrightarrow{AB}$. ✓

Solution

$\overrightarrow{AB} = (3, 1)$, so $\overrightarrow{AM} = \frac{1}{2}(3, 1) = (1.5,\ 0.5)$. Then $M = A + \overrightarrow{AM} = (1, 1) + (1.5, 0.5) = (2.5,\ 1.5)$.
(This point M is the **midpoint** of segment AB — a preview of an idea you'll see formalized later.)

Section 5: The Chasles Relation

Solution

$\overrightarrow{AB} = (3, -3), \quad \overrightarrow{BC} = (2, 4), \quad \overrightarrow{AC} = (5, 1)$.
Check: $\overrightarrow{AB} + \overrightarrow{BC} = (3, -3) + (2, 4) = (5, 1) = \overrightarrow{AC}$. ✓

Solution

By Chasles, $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR} = (2, -5) + (-1, 3) = (1, -2)$.

Solution

Set $C = A$ in $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$: this gives $\overrightarrow{AB} + \overrightarrow{BA} = \overrightarrow{AA} = \mathbf{0}$. Subtracting $\overrightarrow{AB}$ from both sides, $\overrightarrow{BA} = -\overrightarrow{AB}$.

Solution

$\overrightarrow{AB} = (2, 1)$, $\overrightarrow{BC} = (3, 1)$, $\overrightarrow{CD} = (1, 3)$, $\overrightarrow{AD} = (6, 5)$.
Check: $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = (2, 1) + (3, 1) + (1, 3) = (6, 5) = \overrightarrow{AD}$. ✓

Solution

Starting from $\overrightarrow{AB} = \overrightarrow{CD}$, negate both sides: $-\overrightarrow{AB} = -\overrightarrow{CD}$. By the identity from an earlier exercise, $-\overrightarrow{AB} = \overrightarrow{BA}$ and $-\overrightarrow{CD} = \overrightarrow{DC}$. So $\overrightarrow{BA} = \overrightarrow{DC}$.

Solution

$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EA}$. Combine step by step using Chasles: $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$; then $\overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$; then $\overrightarrow{AD} + \overrightarrow{DE} = \overrightarrow{AE}$; and finally $\overrightarrow{AE} + \overrightarrow{EA} = \overrightarrow{AA} = \mathbf{0}$.
So the sum is $\mathbf{0}$ — this always happens for a closed path: walking all the way around and back to where you started gives zero total displacement, no matter the shape of the path or how many vertices it has.

Solution

(a) $\overrightarrow{WZ} = \overrightarrow{WY} + \overrightarrow{YZ}$ (Chasles, one step through Y).

(b) $\overrightarrow{WZ} = \overrightarrow{WX} + \overrightarrow{XY} + \overrightarrow{YZ}$ (Chasles, two steps through X then Y).

(c) From (b), $\overrightarrow{YZ} = \overrightarrow{WZ} - \overrightarrow{WX} - \overrightarrow{XY} = (1, 5) - (3, 1) - (1, 3) = (-3, 1)$.

(d) Setting the right-hand sides of (a) and (b) equal (since both equal $\overrightarrow{WZ}$) and subtracting $\overrightarrow{YZ}$ from both sides gives exactly the equation to prove: $\overrightarrow{WY} = \overrightarrow{WX} + \overrightarrow{XY}$. Expanding every displacement as (head) − (tail):

$$\text{LHS: } \overrightarrow{WX} + \overrightarrow{XY} = (X - W) + (Y - X) = Y - W.$$

$$\text{RHS: } \overrightarrow{WY} = Y - W.$$

Both sides equal $Y - W$, so they're equal for *any* choice of points W, X, Y — the X's cancel algebraically, which is exactly why the intermediate point never matters.

Section 6: Equal, Collinear, and Parallel Vectors

Solution

(a) **Parallel to $\overrightarrow{AB}$:** $\overrightarrow{CD}$, $\overrightarrow{EF}$, $\overrightarrow{GH}$, $\overrightarrow{IJ}$ (all point along the same or the opposite direction as $\overrightarrow{AB}$; $\overrightarrow{KL}$ points a different way and is not parallel).

(b) **Equal to $\overrightarrow{AB}$:** only $\overrightarrow{CD}$ (same length *and* same orientation).

(c) **Opposite to $\overrightarrow{AB}$:** only $\overrightarrow{GH}$ (same length, reversed orientation). $\overrightarrow{EF}$ and $\overrightarrow{IJ}$ are parallel but a different length, so they are collinear with $\overrightarrow{AB}$ without being exactly opposite to it.

Solution

(a) $\mathbf{v} = -3\mathbf{u}$: collinear, opposite orientation, **not** equal.

(b) Neither: $\frac{2}{1} = 2$ but $\frac{7}{4} = 1.75$, so $\mathbf{v} \neq \lambda\mathbf{u}$ for any λ — not collinear.

(c) Equal (same components).

(d) $\mathbf{v} = -2\mathbf{u}$: collinear, opposite orientation, **not** equal.

Solution

Collinear means $(k, -3) = \lambda(4, 6)$ for some λ. From the second component, $-3 = 6\lambda \Rightarrow \lambda = -\frac{1}{2}$. Then $k = 4\lambda = -2$.

Solution

$\overrightarrow{AB} = (2, 4)$ and $\overrightarrow{AC} = (5, 10)$. Since $(5, 10) = \frac{5}{2}(2, 4)$, the vectors are collinear ($\overrightarrow{AC} = \frac{5}{2}\overrightarrow{AB}$). So **yes**, A, B, C are collinear.

Solution

$\overrightarrow{AB} = (2, 3)$ and $\overrightarrow{AC} = (5, 6)$. If $\overrightarrow{AC} = \lambda\overrightarrow{AB}$, then $5 = 2\lambda \Rightarrow \lambda = 2.5$ from the first component, but $6 = 3\lambda \Rightarrow \lambda = 2$ from the second — these disagree, so no single λ works. The vectors are **not** collinear, so A, B, C are **not** collinear.

Section 7: Parallelograms and Parallelepipeds

Solution

Since $\overrightarrow{AD} = \overrightarrow{BC}$, we get $D = A + (C - B) = (0, 0) + ((6, 5) - (4, 1)) = (2, 4)$.
Check: $\overrightarrow{AB} = (4, 1)$ and $\overrightarrow{DC} = (6 - 2, 5 - 4) = (4, 1)$. ✓

Solution

$\overrightarrow{AB} = (5 - 1, 2 - 1) = (4, 1)$ and $\overrightarrow{DC} = (6 - 2, 6 - 5) = (4, 1)$. ✓

$\overrightarrow{AD} = (2-1, 5-1) = (1,4)$ and $\overrightarrow{BC} = (6-5, 6-2) = (1,4)$. ✓
Both pairs of opposite sides match, so $ABCD$ is a parallelogram.

Solution

$G = A + \mathbf{u} + \mathbf{v} + \mathbf{w} = (1,1,0) + (2,0,0) + (0,3,0) + (0,0,4) = (3,4,4)$.

Solution

Since $\overrightarrow{AB} = \overrightarrow{DC}$, we get $C = D + \overrightarrow{AB} = (0,3) + (3,2) = (3,5)$, using $\overrightarrow{AB} = B - A = (3,2)$.
Check: $\overrightarrow{AD} = (-2,4)$ and $\overrightarrow{BC} = (3-5, 5-1) = (-2,4)$. ✓

Solution

$A + \mathbf{u} = (1,2,0), \quad A + \mathbf{v} = (0,1,3), \quad A + \mathbf{w} = (2,0,1)$.
$G = A + \mathbf{u} + \mathbf{v} + \mathbf{w} = (1,2,0) + (0,1,3) + (2,0,1) = (3,3,4)$.

Solution

(a) The parallelogram property $\overrightarrow{AB} = \overrightarrow{DC}$ means $B - A = C - D$. Adding $A + D$ to both sides and rearranging:

$$B - A = C - D \iff B + D = A + C \iff \tfrac{1}{2}(B + D) = \tfrac{1}{2}(A + C) = E.$$

So $E = \frac{1}{2}(B + D)$ too — E is the midpoint of *both* diagonals, which is exactly why the two diagonals cross at a single marked point.
(b)

(i) $\overrightarrow{AE} + \overrightarrow{EB} = \overrightarrow{AB}$ (Chasles).

(ii) Since $\overrightarrow{AD} = \overrightarrow{BC}$, Chasles through A gives $\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD} = \overrightarrow{BA} + \overrightarrow{BC}$, so $\overrightarrow{BC} + \overrightarrow{BA} = \overrightarrow{BD}$.

(iii) $\overrightarrow{AE} + \overrightarrow{AE} = 2\overrightarrow{AE} = 2(E - A) = 2\big(\tfrac{1}{2}(A + C) - A\big) = C - A = \overrightarrow{AC}$.

(iv) $\overrightarrow{AB} - \overrightarrow{DB} = \overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$ (Chasles, using $-\overrightarrow{DB} = \overrightarrow{BD}$).

Section 8: Synthesis and Challenge Problems

Solution

$2\mathbf{u} = (2, -4, 6), \quad 3\mathbf{v} = (0, 12, -3)$.
$2\mathbf{u} - 3\mathbf{v} = (2 - 0,\ -4 - 12,\ 6 - (-3)) = (2, -16, 9)$.

Solution

Both $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are scalar multiples of the same vector $\mathbf{w}$, so they are collinear: they point along the same line through A. Since B lies on the line through A in direction $\mathbf{w}$, and C does too, all three points A, B, C lie on that one line.

Solution

Vector addition is defined component by component: $\mathbf{u} + \mathbf{v} = (u_1 + v_1, \ldots, u_n + v_n)$, which requires $\mathbf{u}$ and $\mathbf{v}$ to have the *same number of components* so that each u_i has a matching v_i to add. A vector in $\mathbb{R}^2$ has 2 components and one in $\mathbb{R}^3$ has 3; there is no third component of the $\mathbb{R}^2$ vector to pair with the third component of the $\mathbb{R}^3$ vector, so the definition simply does not apply.

Solution

Write A, B, C, D for the position vectors (coordinates) of the four points. The hypothesis $\overrightarrow{AB} = \overrightarrow{DC}$ means

$$B - A = C - D.$$

Rearranging (adding $A + D$ to both sides and subtracting A, i.e. moving terms around):

$$B - A = C - D \iff D - A = C - B \iff \overrightarrow{AD} = \overrightarrow{BC}.$$

So the single condition $\overrightarrow{AB} = \overrightarrow{DC}$ already forces $\overrightarrow{AD} = \overrightarrow{BC}$ — one pair of opposite sides being equal (as vectors, in the right order) is enough to guarantee $ABCD$ is a parallelogram.

Solution

Write $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$. The hypothesis $\mathbf{u} + \mathbf{v} = \mathbf{0}$ means, component by component,

$$u_i + v_i = 0 \qquad \text{for every } i = 1, \ldots, n.$$

So $v_i = -u_i$ for every i, which is exactly the statement $\mathbf{v} = (-u_1, \ldots, -u_n) = -\mathbf{u}$.

Solution

Collinear means $\mathbf{v} = \lambda\mathbf{u}$ for some scalar λ, i.e. $k = 2\lambda$ and $8 = k\lambda$. From the first equation $\lambda = \frac{k}{2}$; substituting into the second gives $8 = k \cdot \frac{k}{2} = \frac{k^2}{2}$, so $k^2 = 16$ and $k = 4$ or $k = -4$. Check $k = 4$: $\mathbf{u} = (2, 4)$, $\mathbf{v} = (4, 8) = 2\mathbf{u}$. ✓ Check $k = -4$: $\mathbf{u} = (2, -4)$, $\mathbf{v} = (-4, 8) = -2\mathbf{u}$. ✓