Chapter 2 Exercises

Vector Operations

Norm, Distance, Dot Product, Projection, Cross Product, Scalar Triple Product

Collège Sainte-Anne | Fall 2026

Reference Formulas

$$\|\vec{\mathbf{v}}\| = \sqrt{v_1^2 + \cdots + v_n^2} \qquad \|c\vec{\mathbf{v}}\| = |c|\,\|\vec{\mathbf{v}}\|$$

$$d(A,B) = \|B - A\| \qquad \hat{\vec{\mathbf{v}}} = \frac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|}$$

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1v_1 + \cdots + u_nv_n = \|\vec{\mathbf{u}}\|\,\|\vec{\mathbf{v}}\| \cos\theta$$

$$\mathrm{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{u}}) = \frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|^2}\,\vec{\mathbf{v}}$$

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1)$$

$$\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\| = \|\vec{\mathbf{u}}\|\,\|\vec{\mathbf{v}}\| \sin\theta$$

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Norm and Scaling

Exercise 1.1

Compute the norm of each vector.

(a) $\vec{\mathbf{u}} = (5, 12)$

(b) $\vec{\mathbf{v}} = (2, -3, 6)$

(c) $\vec{\mathbf{w}} = (1, 1, -1, 1)$

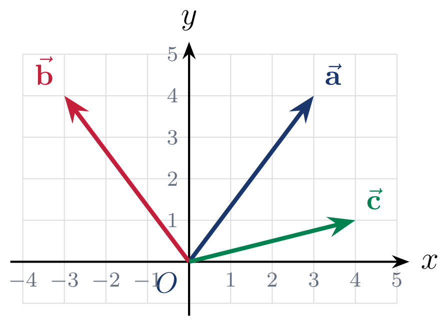
Exercise 1.2

Find all values of k such that $\|(k, 5)\| = 13$.

Hint: Square both sides to remove the square root before solving.

Exercise 1.3

The grid below shows three vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$ drawn from the origin.



(a) Read off the components of $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$ from the graph.

(b) Compute $\|\vec{\mathbf{a}}\|$, $\left\|\vec{\mathbf{b}}\right\|$, $\|\vec{\mathbf{c}}\|$.

(c) Compute the angle θ between the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$?

Exercise 1.4

You are told that $\|\vec{\mathbf{v}}\| = 4$. Without knowing the components of $\vec{\mathbf{v}}$, compute:

(a) $\|3\vec{\mathbf{v}}\|$

(b) $\|-5\vec{\mathbf{v}}\|$

(c) $\left\|\frac{1}{4}\vec{\mathbf{v}}\right\|$

(d) Find a scalar c such that $\|c\vec{\mathbf{v}}\| = 1$. What is the geometric meaning of the vector $c\vec{\mathbf{v}}$?

Exercise 1.5

Find all vectors of the form $(k, 2k)$ in $\mathbb{R}^2$ that have norm $3\sqrt{5}$.

Hint: Substitute into the norm formula and solve for k. There may be more than one answer.

Distance Between Points

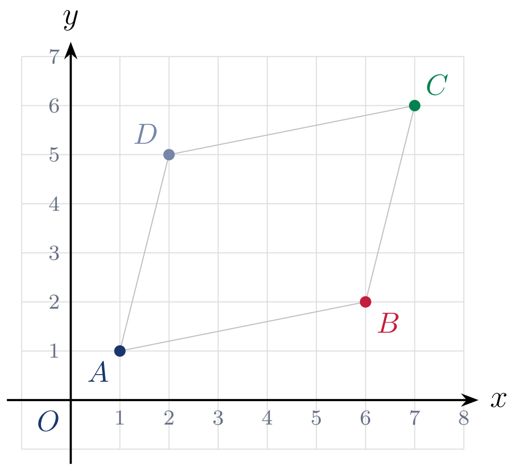
Exercise 2.1

(a) $A = (1, 2, 3)$, $B = (4, 6, 3)$. Compute $d(A, B)$.

(b) $P = (1, k, -2)$, and $d(P, A) = 5$. Find all possible values of k.

Exercise 2.2 — Classifying a quadrilateral from distances

The grid below shows four points A, B, C, D.



(a) Compute the four side lengths $d(A, B)$, $d(B, C)$, $d(C, D)$, $d(D, A)$. What do you notice about the two pairs of opposite sides?

(b) Prove that if $\overrightarrow{AB} = \overrightarrow{DC}$, then $\overrightarrow{AD} = \overrightarrow{BC}$.

(c) Compute the vectors $\overrightarrow{AB}$ and $\overrightarrow{DC}$ to settle whether $ABCD$ is genuinely a parallelogram.

(d) Compute the two diagonal lengths $d(A, C)$ and $d(B, D)$. Without computing anything else, what would it mean for $ABCD$ if these came out equal?

(e) Let E be the point defined by $\overrightarrow{AE} = \overrightarrow{EC}$. Prove that $\overrightarrow{DE} = \overrightarrow{EB}$.

Unit Vectors and Normalization

Exercise 3.1

Find the unit vector in the direction of each vector.

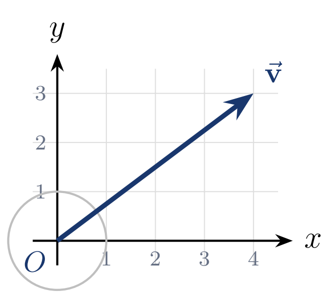
(a) $\vec{\mathbf{u}} = (2, -1, 2)$

(b) $\vec{\mathbf{v}} = (1, 2, 2)$

Verify your answers by computing the norm of each result.

Exercise 3.2

The graph below shows a vector $\vec{\mathbf{v}}$.



(a) Read off $\vec{\mathbf{v}}$ and compute $\|\vec{\mathbf{v}}\|$.

(b) Find $\hat{\vec{\mathbf{v}}}$ and draw it on the graph.

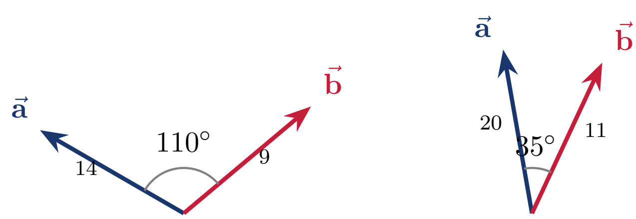
(c) The circle shown has radius 1. Where does $\hat{\vec{\mathbf{v}}}$ land on that circle? Give exact coordinates.

(d) A classmate normalizes $\vec{\mathbf{v}}$ by dividing by the *sum* of its components instead of its norm, getting $\left(\frac{4}{7}, \frac{3}{7}\right)$. Show this is not a unit vector, and explain what step of the normalization process they skipped.

The Dot Product and Angles

Exercise 4.1

For each pair below, θ is the angle between the vectors when placed tail to tail. Compute $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$ using $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \|\vec{\mathbf{a}}\| \left\|\vec{\mathbf{b}}\right\| \cos\theta$.



(a) (b)

Exercise 4.2

For each pair of vectors below, first **predict the sign** of the dot product (positive, zero, or negative) by sketching the vectors roughly, then compute it to verify.

(a) $\vec{\mathbf{u}} = (3, 1)$ and $\vec{\mathbf{v}} = (1, 3)$

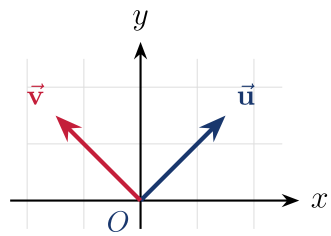
(b) $\vec{\mathbf{u}} = (2, 3, -1)$ and $\vec{\mathbf{v}} = (-3, 1, -1)$

(c) $\vec{\mathbf{u}} = (1, 1)$ and $\vec{\mathbf{v}} = (-1, 1)$

Exercise 4.3

(a) Find the angle between $\vec{\mathbf{u}} = (1, 0, 0)$ and $\vec{\mathbf{v}} = (1, 1, 0)$.

(b) Find the angle between $\vec{\mathbf{u}} = (1, 1)$ and $\vec{\mathbf{v}} = (-1, 1)$. Verify your answer makes sense from the graph below.



Exercise 4.4

A triangle has vertices $A = (1, 1)$, $B = (2, 4)$, $C = (5, 3)$.

(a) At each vertex, form the two vectors pointing away from it along the two sides meeting there (e.g. at A: $\overrightarrow{AB}$ and $\overrightarrow{AC}$).

(b) Compute the three dot products needed to test each vertex.

(c) Does this triangle have a right angle? If so, at which vertex?

Exercise 4.5

A triangle DEF has vertices $D = (-2, 3)$, $E = (3, 1)$, $F = (6, 3)$.

(a) To compute the interior angle at E, you need two vectors that both start *at* E. Which two vectors are they?

(b) Write down those two vectors and use them to compute the angle at E.

(c) Now compute the angle between $\overrightarrow{DE}$ and $\overrightarrow{EF}$ instead (note that $\overrightarrow{DE}$ points *into* E, not out of it, so the two vectors no longer share a common start point). Compare this angle to your answer in part (b). What do you notice, and why does mixing directions like this happen to produce that result?

(d) Using the same reasoning as in part (a), identify the two vectors needed to compute the interior angle at vertex D. You do not need to compute the numeric value of the angle (just name the two vectors).

Exercise 4.6

Find the value of $k > 0$ such that the angle between $\vec{\mathbf{u}} = (k, 3)$ and $\vec{\mathbf{v}} = (1, 0)$ is $60°$.

Exercise 4.7

For each statement, decide **True** or **False** and justify your answer with either a proof or a counterexample.

(a) If $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$ and $\vec{\mathbf{v}} \cdot \vec{\mathbf{w}} = 0$, then $\vec{\mathbf{u}} \cdot \vec{\mathbf{w}} = 0$.

(b) If $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} > 0$ and $\vec{\mathbf{v}} \cdot \vec{\mathbf{w}} > 0$, then $\vec{\mathbf{u}} \cdot \vec{\mathbf{w}} > 0$.

(c) $\|\vec{\mathbf{u}}+\vec{\mathbf{v}}\|^2 = \|\vec{\mathbf{u}}\|^2 + \|\vec{\mathbf{v}}\|^2$ whenever $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are perpendicular.

Exercise 4.8

A triangle has vertices $P = (2, 1, 3)$, $Q = (2, 1, 8)$, $R = (5, 5, 3)$.

(a) Form the three side vectors and compute their norms to find the three side lengths.

(b) Based on the side lengths alone, is the triangle scalene, isosceles, or equilateral?

(c) Use the dot product to test the angle at each vertex. Does the triangle have a right angle?

Projection and Orthogonality

Exercise 5.1

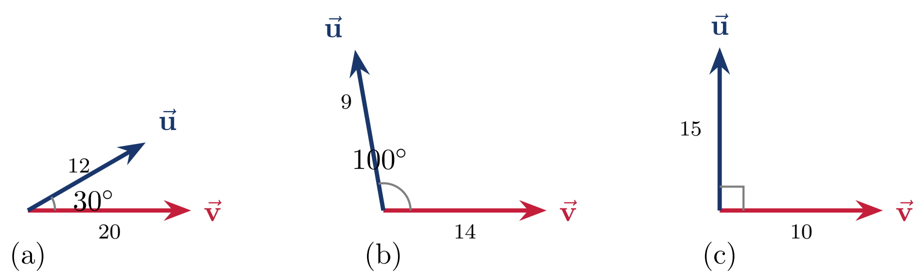
(a) Find $\text{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{u}})$ for $\vec{\mathbf{u}} = (4, 2)$ and $\vec{\mathbf{v}} = (1, 0)$.

(b) Find $\text{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{u}})$ for $\vec{\mathbf{u}} = (3, 1, 2)$ and $\vec{\mathbf{v}} = (-2, 0, 0)$.

Without computing, explain why both answers should lie along the x-axis. Then verify.

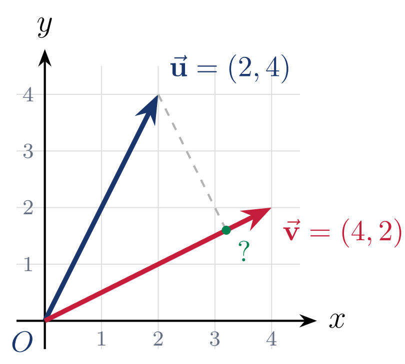
Exercise 5.2

For each pair, draw the projection $\vec{\mathbf{u}}$ on $\vec{\mathbf{v}}$ ($\text{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{u}})$) and find its magnitude, then say whether the projection points in the same direction as $\vec{\mathbf{v}}$, the opposite direction, or is the zero vector.



Exercise 5.3

The grid below shows vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.



(a) The dashed line drops perpendicularly from the tip of $\vec{\mathbf{u}}$ to the line of $\vec{\mathbf{v}}$. The green dot marks the foot of this perpendicular. Compute $\mathrm{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{u}})$ and verify it matches the dot.

(b) Compute the perpendicular component $\vec{\mathbf{u}}_\perp = \vec{\mathbf{u}} - \mathrm{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{u}})$.

(c) Verify that $\vec{\mathbf{u}}_\perp \cdot \vec{\mathbf{v}} = 0$.

Exercise 5.4

A crate is pushed across a floor by a force $\vec{\mathbf{F}} = (3, 4)$ newtons through a displacement $\vec{\mathbf{d}} = (8, 6)$ meters (the crate is dragged diagonally, not straight along an axis).

(a) Compute the work done $W = \vec{\mathbf{F}} \cdot \vec{\mathbf{d}}$.

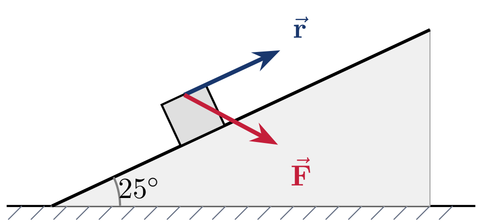
(b) Find the component of $\vec{\mathbf{F}}$ that is *parallel* to the direction of motion (the effective force).

(c) Find the component of $\vec{\mathbf{F}}$ that is *perpendicular* to the motion (the wasted force pressing into the floor).

(d) Verify that the two components add back to $\vec{\mathbf{F}}$.

Exercise 5.5

A crate is pushed up a straight ramp. The ramp's up-slope direction is given by the unit vector $\vec{\mathbf{r}} = (\cos 25°, \sin 25°)$, and the crate slides 6 m along the ramp. The worker applies a (non-aligned) force $\vec{\mathbf{F}} = (150, -80)$ N.



(a) Write the displacement vector $\vec{\mathbf{d}} = 6\vec{\mathbf{r}}$ (components to two decimals).

(b) Compute the work done, $W = \vec{\mathbf{F}} \cdot \vec{\mathbf{d}}$.

(c) Find $\mathrm{proj}_{\vec{\mathbf{r}}}(\vec{\mathbf{F}})$, the component of the force that actually acts along the ramp.

(d) Find the component of $\vec{\mathbf{F}}$ perpendicular to the ramp. Physically, what must supply an equal and opposite force to this component?

Exercise 5.6

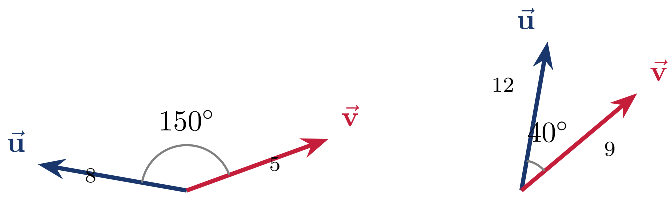
(a) Find k such that $\vec{\mathbf{u}} = (k, 2, -3)$ is orthogonal to $\vec{\mathbf{v}} = (1, -4, 2)$.

(b) Find all vectors of the form $(a, b, 0)$ with $\|(a, b, 0)\| = 1$ that are orthogonal to $\vec{\mathbf{w}} = (1, 1, 0)$.

The Cross Product

Exercise 6.1

For each pair, find $\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|$ using $\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\| = \|\vec{\mathbf{u}}\|\,\|\vec{\mathbf{v}}\|\sin\theta$.



(a) (b)

Exercise 6.2

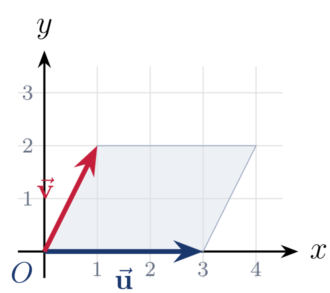
Compute each cross product and verify the result is orthogonal to both inputs by checking the two dot products.

(a) $\vec{\mathbf{u}} = (1, 0, 2) \times \vec{\mathbf{v}} = (0, 1, 3)$

(b) $\vec{\mathbf{u}} = (2, 1, 0) \times \vec{\mathbf{v}} = (0, 0, 1)$

Exercise 6.3

The parallelogram below is spanned by vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ drawn from the origin.



(a) Read the components of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ from the graph.

(b) Without computing, estimate the area of the parallelogram by comparing it to the unit squares on the grid.

(c) Compute the exact area using the cross product. Was your estimate close?

Exercise 6.4

Four points in space are given: $A = (1, -2, 0)$, $B = (4, -1, 3)$, $C = (6, 3, 3)$, $D = (3, 2, 0)$.

(a) Compute $\overrightarrow{AB}$ and $\overrightarrow{DC}$ and verify that the quadrilateral $ABCD$ is a parallelogram ?

(b) Compute the area of parallelogram $ABCD$.

Exercise 6.5

Three points are given: $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$.

(a) Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

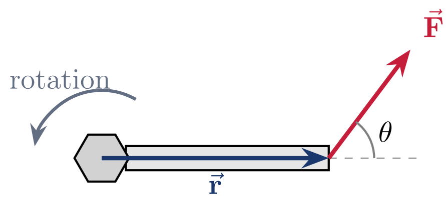
(b) Compute $\overrightarrow{AB} \times \overrightarrow{AC}$.

(c) What is the geometric meaning of your result? Normalise it.

(d) Find the area of triangle ABC.

Exercise 6.6

A wrench has a handle of length $\|\vec{\mathbf{r}}\| = 0.3$ m. A force $\vec{\mathbf{F}}$ of magnitude $\left\|\vec{\mathbf{F}}\right\| = 20$ N is applied at the end.



(a) If $\vec{\mathbf{r}} = (0.3, 0, 0)$ and $\vec{\mathbf{F}} = (12, 16, 0)$ (still magnitude 20 N, but not perpendicular to the handle), compute the torque $\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$ and its magnitude.

(b) Using the magnitude formula $\left\|\vec{\mathbf{r}} \times \vec{\mathbf{F}}\right\| = \|\vec{\mathbf{r}}\| \left\|\vec{\mathbf{F}}\right\| \sin\theta$, find the angle θ between $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$ in part (a). Does your answer agree with the geometry?

(c) What angle between the handle and the force produces the maximum torque? What is that maximum value?

(d) A second scenario has $\vec{\mathbf{r}} = (0.3, 0, 0)$ and $\vec{\mathbf{F}} = (20, 0, 0)$ (force along the handle). Compute the torque. Explain physically why pushing along the handle produces no rotation.

Collinearity and Coplanarity

Exercise 7.1

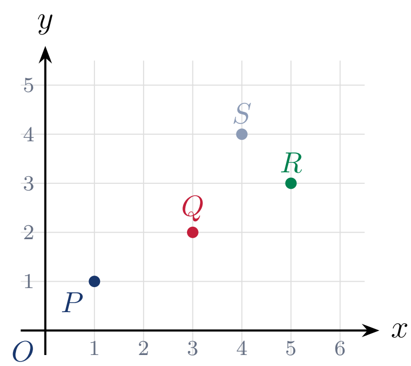
For each pair, determine whether the vectors are collinear. Show your work using the cross product; also verify by inspection where possible.

(a) $\vec{\mathbf{u}} = (3, -6, 9)$ and $\vec{\mathbf{v}} = (-1, 2, -3)$

(b) $\vec{\mathbf{u}} = (1, 2, 3)$ and $\vec{\mathbf{v}} = (2, 4, 5)$

Exercise 7.2

The figure below shows four points on a grid in $\mathbb{R}^2$.



(a) By inspection, do P, Q, R appear to be collinear?

(b) Confirm algebraically using vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ (embed in $\mathbb{R}^3$ with $z = 0$).

(c) Does S lie on the same line? Check algebraically.

Scalar Triple Product

Exercise 8.1

Compute the volume of the parallelepiped spanned by:

(a) $\vec{\mathbf{u}} = (2, 0, 0)$, $\vec{\mathbf{v}} = (0, 3, 0)$, $\vec{\mathbf{w}} = (0, 0, 5)$. Before computing, predict the answer from the geometry.

(b) $\vec{\mathbf{u}} = (1, 2, 0)$, $\vec{\mathbf{v}} = (0, 1, 3)$, $\vec{\mathbf{w}} = (2, 0, 1)$.

Exercise 8.2

Determine whether each set of vectors is coplanar.

(a) $\vec{\mathbf{u}} = (1, 0, 1)$, $\vec{\mathbf{v}} = (0, 1, 1)$, $\vec{\mathbf{w}} = (1, 1, 2)$.

(b) $\vec{\mathbf{u}} = (1, 1, 0)$, $\vec{\mathbf{v}} = (0, 1, 1)$, $\vec{\mathbf{w}} = (1, 0, 1)$.

For each case that is coplanar, explain *why* geometrically.

Exercise 8.3

Let $\vec{\mathbf{u}} = (1, 0, 1)$, $\vec{\mathbf{v}} = (0, 1, 0)$, $\vec{\mathbf{w}} = (1, 1, 0)$.

(a) Compute $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$. Are $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ orthogonal?

(b) Compute $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$.

(c) Compute $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})$ using the determinant formula.

(d) Are $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, $\vec{\mathbf{w}}$ coplanar? Justify.

(e) Using $\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\| = \|\vec{\mathbf{u}}\|\,\|\vec{\mathbf{v}}\| \sin\theta$, find the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.

(f) Verify your answer in (e) using the dot product formula instead. Do both methods agree?

Exercise 8.4

Let $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, $\vec{\mathbf{w}}$, $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ be vectors in $\mathbb{R}^3$. Simplify each expression as far as possible.

(a) $\vec{\mathbf{u}} \cdot \big[(\vec{\mathbf{u}} + \vec{\mathbf{v}}) \times (\vec{\mathbf{u}} - \vec{\mathbf{v}})\big] + 2\,\vec{\mathbf{w}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) - \vec{\mathbf{w}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{u}})$

(b) $\vec{\mathbf{a}} \times (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) + \|\vec{\mathbf{a}}\|^2\,\vec{\mathbf{b}}$ (use the **BAC-CAB** rule : $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})\vec{\mathbf{b}} - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\vec{\mathbf{c}}$)

(c) $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) - \vec{\mathbf{v}} \cdot (\vec{\mathbf{w}} \times \vec{\mathbf{u}}) + \vec{\mathbf{w}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) - 3\,\vec{\mathbf{w}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{u}})$

(d) $\big[(\vec{\mathbf{u}} + \vec{\mathbf{v}}) \times (\vec{\mathbf{u}} - \vec{\mathbf{v}})\big] \cdot \vec{\mathbf{w}} \;+\; 2\,\vec{\mathbf{w}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{u}}) \;-\; \vec{\mathbf{u}} \cdot \big[\vec{\mathbf{u}} \times (\vec{\mathbf{v}} + \vec{\mathbf{w}})\big] \;+\; \dfrac{\|\vec{\mathbf{u}}\|^2}{3} - \vec{\mathbf{u}} \cdot \vec{\mathbf{u}}$

(e) $\big[\vec{\mathbf{u}} \times (\vec{\mathbf{v}} \times \vec{\mathbf{w}})\big] \cdot \vec{\mathbf{u}} \;+\; \big[\vec{\mathbf{v}} \times (\vec{\mathbf{w}} \times \vec{\mathbf{u}})\big] \cdot \vec{\mathbf{v}} \;+\; \big[\vec{\mathbf{w}} \times (\vec{\mathbf{u}} \times \vec{\mathbf{v}})\big] \cdot \vec{\mathbf{w}}$

(f) $\vec{\mathbf{u}} \times (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) + \vec{\mathbf{v}} \times (\vec{\mathbf{w}} \times \vec{\mathbf{u}}) + \vec{\mathbf{w}} \times (\vec{\mathbf{u}} \times \vec{\mathbf{v}})$

(g) $2\,\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) + \vec{\mathbf{v}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{w}}) - \vec{\mathbf{w}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{u}}) - 4\,\vec{\mathbf{v}} \cdot (\vec{\mathbf{w}} \times \vec{\mathbf{u}}) + 3\,\vec{\mathbf{w}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}})$

(h) $2\,\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) + 3\,\vec{\mathbf{b}} \cdot (\vec{\mathbf{a}} \times \vec{\mathbf{c}}) - \vec{\mathbf{c}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{a}}) + \vec{\mathbf{a}} \cdot (\vec{\mathbf{c}} \times \vec{\mathbf{b}})$