Chapter 2 Exercises

Solutions

Vector Operations

Norm, Distance, Dot Product, Projection, Cross Product, Scalar Triple Product

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Norm and Scaling

Exercise 1.1

(a) $\vec{u} = (5, 12)$: $\|\vec{u}\| = \sqrt{5^2 + 12^2} = \sqrt{169} = \boxed{13}$.

(b) $\vec{v} = (2, -3, 6)$: $\|\vec{v}\| = \sqrt{4 + 9 + 36} = \sqrt{49} = \boxed{7}$.

(c) $\vec{w} = (1, 1, -1, 1)$: $\|\vec{w}\| = \sqrt{1 + 1 + 1 + 1} = \sqrt{4} = \boxed{2}$.

Exercise 1.2

Squaring $\|(k, 5)\| = 13$ gives $k^2 + 25 = 169$, so $k^2 = 144$ and

$$\boxed{k = \pm 12}.$$

Exercise 1.3

(a) Reading the tips of the arrows off the grid: $\vec{a} = (3, 4)$, $\vec{b} = (-3, 4)$, $\vec{c} = (4, 1)$.

(b) $\|\vec{a}\| = \sqrt{9 + 16} = 5$, $\left\|\vec{b}\right\| = \sqrt{9 + 16} = 5$, $\|\vec{c}\| = \sqrt{16 + 1} = \sqrt{17}$.

(c) $\vec{a} \cdot \vec{b} = (3)(-3) + (4)(4) = 7$, so

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \left\|\vec{b}\right\|} = \frac{7}{5 \cdot 5} = \frac{7}{25} \implies \boxed{\theta = \arccos(0.28) \approx 73.7^\circ}.$$

Exercise 1.4

Given $\|\vec{v}\| = 4$:

(a) $\|3\vec{v}\| = 3\,\|\vec{v}\| = \boxed{12}$.

(b) $\|-5\vec{v}\| = |-5|\,\|\vec{v}\| = \boxed{20}$.

(c) $\left\|\frac{1}{4}\vec{v}\right\| = \frac{1}{4} \cdot 4 = \boxed{1}$.

(d) We need $|c| \cdot 4 = 1$, i.e. $|c| = \frac{1}{4}$, so $c = \pm\frac{1}{4}$. Geometrically, $c\vec{v}$ is a *unit vector* in the direction of $\vec{v}$ (for $c = \frac{1}{4}$) or in the opposite direction (for $c = -\frac{1}{4}$) – rescaling $\vec{v}$ down to length 1.

Exercise 1.5

Substituting into the norm formula: $\|(k, 2k)\| = 3\sqrt{5}$ gives

$$k^2 + 4k^2 = 45 \implies 5k^2 = 45 \implies k^2 = 9 \implies k = \pm 3.$$

So the vectors are $\boxed{(3, 6) \text{ and } (-3, -6)}$.

Distance Between Points

Exercise 2.1

(a) $d(A, B) = \|B - A\| = \|(3, 4, 0)\| = \sqrt{9 + 16} = \boxed{5}$.

(b) $d(P, A)^2 = (1 - 1)^2 + (k - 2)^2 + (-2 - 3)^2 = (k - 2)^2 + 25$. Setting this equal to 25 gives $(k - 2)^2 = 0$, so $\boxed{k = 2}$ is the *only* solution – the z-difference alone already accounts for the full distance of 5.

Exercise 2.2

Here $A = (1, 1)$, $B = (6, 2)$, $C = (7, 6)$, $D = (2, 5)$.

(a) $\overrightarrow{AB} = (5, 1)$, $d(A, B) = \sqrt{26}$; $\overrightarrow{BC} = (1, 4)$, $d(B, C) = \sqrt{17}$; $\overrightarrow{CD} = (-5, -1)$, $d(C, D) = \sqrt{26}$; $\overrightarrow{DA} = (-1, -4)$, $d(D, A) = \sqrt{17}$.
The two pairs of opposite sides are equal in length: $d(A, B) = d(C, D) = \sqrt{26}$ and $d(B, C) = d(D, A) = \sqrt{17}$.

(b) If $\overrightarrow{AB} = \overrightarrow{DC}$, then $B - A = C - D$. Rearranging, $D - A = C - B$, i.e. $\overrightarrow{AD} = \overrightarrow{BC}$.

(c) $\overrightarrow{AB} = (5, 1)$ and $\overrightarrow{DC} = C - D = (5, 1)$. Since $\overrightarrow{AB} = \overrightarrow{DC}$ exactly (not just equal in length), $ABCD$ **is** a genuine parallelogram – equal side lengths alone would not be enough (an isosceles trapezoid also has that property).

(d) $d(A, C) = \|(6, 5)\| = \sqrt{61}$ and $d(B, D) = \|(-4, 3)\| = 5$. Since $\sqrt{61} \neq 5$, the diagonals are unequal. If they *had* come out equal, $ABCD$ would be a **rectangle** (a parallelogram with equal diagonals).

(e) $\overrightarrow{AE} = \overrightarrow{EC}$ means $E - A = C - E$, so $E = \dfrac{A + C}{2} = (4, 3.5)$ – the midpoint of AC. Then

$$\overrightarrow{DE} = E - D = (2, -1.5), \qquad \overrightarrow{EB} = B - E = (2, -1.5),$$

so $\overrightarrow{DE} = \overrightarrow{EB}$. (This is really the statement that the diagonals of a parallelogram bisect each other at the same point E.)

Unit Vectors and Normalization

Exercise 3.1

(a) $\vec{u} = (2, -1, 2)$: $\|\vec{u}\| = \sqrt{4+1+4} = 3$, so $\hat{\vec{u}} = \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$.

Check: $\sqrt{\frac{4}{9} + \frac{1}{9} + \frac{4}{9}} = \sqrt{1} = 1$. ✓

(b) $\vec{v} = (1, 2, 2)$: $\|\vec{v}\| = \sqrt{1+4+4} = 3$, so $\hat{\vec{v}} = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$.

Check: $\sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} = \sqrt{1} = 1$. ✓

Exercise 3.2

(a) From the graph, $\vec{v} = (4, 3)$, so $\|\vec{v}\| = \sqrt{16+9} = 5$.

(b) $\hat{\vec{v}} = \left(\frac{4}{5}, \frac{3}{5}\right)$.

(c) Since $\left\|\hat{\vec{v}}\right\| = 1$ by construction, $\hat{\vec{v}}$ lands exactly on the unit circle, at $\boxed{\left(\frac{4}{5}, \frac{3}{5}\right)}$.

(d) The classmate computed $\left(\frac{4}{7}, \frac{3}{7}\right)$, using the sum of components $4 + 3 = 7$ instead of the norm. Its length is

$$\sqrt{\left(\frac{4}{7}\right)^2 + \left(\frac{3}{7}\right)^2} = \sqrt{\frac{16+9}{49}} = \sqrt{\frac{25}{49}} = \frac{5}{7} \neq 1,$$

so it is **not** a unit vector. It does still point in the same direction as $\vec{v}$ (dividing by 7 is just a positive rescaling), but the classmate skipped the step of dividing by $\sqrt{v_1^2 + v_2^2}$ (the norm) – they used $v_1 + v_2$ instead.

The Dot Product and Angles

Exercise 4.1

(a) $\vec{a} \cdot \vec{b} = \|\vec{a}\| \left\|\vec{b}\right\| \cos\theta = (14)(9)\cos 110^\circ \approx (126)(-0.342) \approx \boxed{-43.1}$.

(b) $\vec{a} \cdot \vec{b} = (20)(11)\cos 35^\circ \approx (220)(0.819) \approx \boxed{180.2}$.

Exercise 4.2

(a) $\vec{u} = (3, 1), \vec{v} = (1, 3)$: both point into the first quadrant at similar angles, so we expect a *positive* dot product. Indeed, $\vec{u} \cdot \vec{v} = 3 + 3 = \boxed{6} > 0$.

(b) $\vec{u} = (2, 3, -1), \vec{v} = (-3, 1, -1)$: the signs are mixed, so the sign is hard to guess from a sketch alone. Computing, $\vec{u} \cdot \vec{v} = -6 + 3 + 1 = \boxed{-2} < 0$.

(c) $\vec{u} = (1,1), \vec{v} = (-1,1)$: $\vec{v}$ looks like $\vec{u}$ rotated a quarter turn, so we expect 0. Indeed $\vec{u} \cdot \vec{v} = -1 + 1 = \boxed{0}$ – they are exactly orthogonal.

Exercise 4.3

(a) $\vec{u} = (1,0,0)$, $\vec{v} = (1,1,0)$: $\cos\theta = \dfrac{1}{1 \cdot \sqrt{2}} = \dfrac{\sqrt{2}}{2}$, so $\boxed{\theta = 45^\circ}$.

(b) $\vec{u} = (1,1)$, $\vec{v} = (-1,1)$: $\cos\theta = \dfrac{-1+1}{\sqrt{2} \cdot \sqrt{2}} = 0$, so $\boxed{\theta = 90^\circ}$ – consistent with the graph, where $\vec{v}$ is $\vec{u}$ rotated by a quarter turn.

Exercise 4.4

Triangle $A = (1,1)$, $B = (2,4)$, $C = (5,3)$.

(a) At A: $\overrightarrow{AB} = (1,3)$, $\overrightarrow{AC} = (4,2)$. At B: $\overrightarrow{BA} = (-1,-3)$, $\overrightarrow{BC} = (3,-1)$. At C: $\overrightarrow{CA} = (-4,-2)$, $\overrightarrow{CB} = (-3,1)$.

(b) $\overrightarrow{AB} \cdot \overrightarrow{AC} = 4 + 6 = 10$; $\quad \overrightarrow{BA} \cdot \overrightarrow{BC} = -3 + 3 = 0$; $\quad \overrightarrow{CA} \cdot \overrightarrow{CB} = 12 - 2 = 10$.

(c) Since $\overrightarrow{BA} \cdot \overrightarrow{BC} = 0$, the triangle has a $\boxed{\text{right angle at } B}$.

Exercise 4.5

Triangle $D = (-2,3)$, $E = (3,1)$, $F = (6,3)$.

(a) The interior angle at E needs the two vectors that *start* at E: $\overrightarrow{ED}$ and $\overrightarrow{EF}$.

(b) $\overrightarrow{ED} = (-5,2)$, $\overrightarrow{EF} = (3,2)$.

$$\cos\theta = \frac{(-5)(3) + (2)(2)}{\sqrt{29}\sqrt{13}} = \frac{-11}{\sqrt{377}} \approx -0.567 \implies \boxed{\theta \approx 124.5^\circ}.$$

(c) $\overrightarrow{DE} = (5,-2)$, $\overrightarrow{EF} = (3,2)$.

$$\cos\theta' = \frac{(5)(3) + (-2)(2)}{\sqrt{29}\sqrt{13}} = \frac{11}{\sqrt{377}} \approx 0.567 \implies \theta' \approx 55.5^\circ.$$

Notice $\theta + \theta' = 180^\circ$: the two angles are **supplementary**. This happens because $\overrightarrow{DE} = -\overrightarrow{ED}$, and reversing one of the two vectors in a dot product flips the sign of $\cos\theta$, replacing an angle by its supplement.

(d) At D, the interior angle needs $\overrightarrow{DE}$ and $\overrightarrow{DF}$.

Exercise 4.6

We need $\cos 60^\circ = \dfrac{k}{\sqrt{k^2+9}} = \dfrac{1}{2}$. Squaring, $4k^2 = k^2 + 9$, so $3k^2 = 9$, $k^2 = 3$, and taking $k > 0$,

$$\boxed{k = \sqrt{3}}.$$

Exercise 4.7

(a) **False.** Counterexample: $\vec{u} = (1, 0)$, $\vec{v} = (0, 1)$, $\vec{w} = (1, 0)$. Then $\vec{u} \cdot \vec{v} = 0$ and $\vec{v} \cdot \vec{w} = 0$, but $\vec{u} \cdot \vec{w} = 1 \neq 0$. Being perpendicular to a common vector does not force $\vec{u}$ and $\vec{w}$ to be perpendicular to each other.

(b) **False.** Counterexample: $\vec{u} = (1, 0)$, $\vec{v} = (1, 1)$, $\vec{w} = (0, 1)$. Then $\vec{u} \cdot \vec{v} = 1 > 0$ and $\vec{v} \cdot \vec{w} = 1 > 0$, but $\vec{u} \cdot \vec{w} = 0$, which is not > 0.

(c) **True.** If $\vec{u} \cdot \vec{v} = 0$, then

$$\|\vec{u} + \vec{v}\|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = \vec{u} \cdot \vec{u} + 2\,\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v} = \|\vec{u}\|^2 + \|\vec{v}\|^2\,,$$

which is exactly the Pythagorean theorem written with vectors.

Exercise 4.8

Triangle $P = (2, 1, 3)$, $Q = (2, 1, 8)$, $R = (5, 5, 3)$.

(a) $\overrightarrow{PQ} = (0, 0, 5)$, $\left\|\overrightarrow{PQ}\right\| = 5$. $\overrightarrow{PR} = (3, 4, 0)$, $\left\|\overrightarrow{PR}\right\| = 5$. $\overrightarrow{QR} = (3, 4, -5)$, $\left\|\overrightarrow{QR}\right\| = \sqrt{9 + 16 + 25} = \sqrt{50} = 5\sqrt{2}$.

(b) Two sides are equal ($PQ = PR = 5$), so the triangle is $\boxed{\text{isosceles}}$.

(c) $\overrightarrow{PQ} \cdot \overrightarrow{PR} = 0 + 0 + 0 = 0$, so the $\boxed{\text{right angle is at } P}$ – it is in fact an isosceles right triangle.

Projection and Orthogonality

Exercise 5.1

(a) $\text{proj}_{\vec{v}}(\vec{u}) = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v} = \dfrac{4}{1}(1, 0) = \boxed{(4, 0)}$.

(b) $\text{proj}_{\vec{v}}(\vec{u}) = \dfrac{-6}{4}(-2, 0, 0) = \boxed{(3, 0, 0)}$.

Both results lie along the x-axis simply because $\vec{v}$ itself points along the x-axis in each case: a projection onto $\vec{v}$ is, by definition, a scalar multiple of $\vec{v}$, regardless of $\vec{u}$. The computations above confirm this.

Exercise 5.2

(a) Signed magnitude $= \|\vec{u}\| \cos 30° = 12 \cdot \dfrac{\sqrt{3}}{2} = 6\sqrt{3} \approx \boxed{10.39}$. Positive $\Rightarrow$ **same direction** as $\vec{v}$.

(b) Signed magnitude $= 9 \cos 100° \approx \boxed{-1.56}$. Negative $\Rightarrow$ **opposite direction** to $\vec{v}$ (the angle is obtuse).

(c) The angle shown is $90°$, so the magnitude is $15 \cos 90° = 0$: the projection is the $\boxed{\text{zero vector}}$.

Exercise 5.3

$\vec{u} = (2, 4)$, $\vec{v} = (4, 2)$.

(a) $\text{proj}_{\vec{v}}(\vec{u}) = \dfrac{8+8}{16+4}(4,2) = \dfrac{16}{20}(4,2) = \left(\dfrac{16}{5}, \dfrac{8}{5}\right)$ – matches the green dot on the graph.

(b) $\vec{u}_\perp = \vec{u} - \text{proj}_{\vec{v}}(\vec{u}) = (2,4) - \left(\dfrac{16}{5}, \dfrac{8}{5}\right) = \left(-\dfrac{6}{5}, \dfrac{12}{5}\right)$.

(c) $\vec{u}_\perp \cdot \vec{v} = \left(-\dfrac{6}{5}\right)(4) + \left(\dfrac{12}{5}\right)(2) = -\dfrac{24}{5} + \dfrac{24}{5} = 0.$ ✓

Exercise 5.4

$\vec{F} = (3,4)$ N, $\vec{d} = (8,6)$ m.

(a) $W = \vec{F} \cdot \vec{d} = 24 + 24 = \boxed{48}$ J.

(b) $\text{proj}_{\vec{d}}(\vec{F}) = \dfrac{48}{100}(8,6) = \left(\dfrac{96}{25}, \dfrac{72}{25}\right) \approx (3.84,\ 2.88)$ N.

(c) $\vec{F}_\perp = \vec{F} - \text{proj}_{\vec{d}}(\vec{F}) = \left(-\dfrac{21}{25}, \dfrac{28}{25}\right) \approx (-0.84,\ 1.12)$ N.

(d) $\left(\dfrac{96}{25} - \dfrac{21}{25}, \dfrac{72}{25} + \dfrac{28}{25}\right) = \left(\dfrac{75}{25}, \dfrac{100}{25}\right) = (3,4) = \vec{F}.$ ✓

Exercise 5.5

$\vec{r} = (\cos 25°, \sin 25°)$.

(a) $\vec{d} = 6\vec{r} \approx \boxed{(5.44,\ 2.54)}$ m.

(b) $W = \vec{F} \cdot \vec{d} \approx 150(5.44) + (-80)(2.54) \approx 816.6 - 203.2 \approx \boxed{612.8}$ J.

(c) Since $\vec{r}$ is a unit vector, the scalar component along the ramp is $\vec{F} \cdot \vec{r} \approx 150 \cos 25° - 80 \sin 25° \approx 135.9 - 33.8 \approx 102.1$ N, so

$$\text{proj}_{\vec{r}}(\vec{F}) = (\vec{F} \cdot \vec{r})\,\vec{r} \approx (102.1)(0.906,\ 0.423) \approx \boxed{(92.6,\ 43.2)} \text{ N}.$$

(d) $\vec{F}_\perp = \vec{F} - \text{proj}_{\vec{r}}(\vec{F}) \approx (57.4, -123.2)$ N, with magnitude ≈ 135.9 N. Physically, this is the component pressing the crate *into* the ramp; the **normal force** from the ramp surface must supply an equal and opposite force to keep the crate from sinking into the incline.

Exercise 5.6

(a) $\vec{u} \cdot \vec{v} = k - 8 - 6 = k - 14 = 0 \implies \boxed{k = 14}$.

(b) Orthogonality to $\vec{w} = (1,1,0)$ requires $a + b = 0$, i.e. $b = -a$. Combined with $a^2 + b^2 = 1$: $2a^2 = 1$, so $a = \pm\dfrac{\sqrt{2}}{2}$. The two vectors are

$$\boxed{\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\right) \text{ and } \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)}.$$

The Cross Product

Exercise 6.1

(a) $\|\vec{u} \times \vec{v}\| = 8 \cdot 5 \cdot \sin 150° = 40 \cdot 0.5 = \boxed{20}$.

(b) $\|\vec{u} \times \vec{v}\| = 12 \cdot 9 \cdot \sin 40° \approx 108 \cdot 0.643 \approx \boxed{69.4}$.

Exercise 6.2

(a) $\vec{u} \times \vec{v} = (1, 0, 2) \times (0, 1, 3) = (0 \cdot 3 - 2 \cdot 1,\ 2 \cdot 0 - 1 \cdot 3,\ 1 \cdot 1 - 0 \cdot 0) = \boxed{(-2, -3, 1)}$.
Check: $\vec{u} \cdot (\vec{u} \times \vec{v}) = -2 + 0 + 2 = 0$ ✓; $\vec{v} \cdot (\vec{u} \times \vec{v}) = 0 - 3 + 3 = 0$ ✓.

(b) $\vec{u} \times \vec{v} = (2, 1, 0) \times (0, 0, 1) = (1 \cdot 1 - 0 \cdot 0,\ 0 \cdot 0 - 2 \cdot 1,\ 2 \cdot 0 - 1 \cdot 0) = \boxed{(1, -2, 0)}$.
Check: $\vec{u} \cdot (\vec{u} \times \vec{v}) = 2 - 2 + 0 = 0$ ✓; $\vec{v} \cdot (\vec{u} \times \vec{v}) = 0$ ✓.

Exercise 6.3

(a) From the graph: $\vec{u} = (3, 0)$, $\vec{v} = (1, 2)$.

(b) The parallelogram sits inside roughly a 3×2 bounding box and covers a good majority of it – a reasonable visual estimate is around 5–6 square units.

(c) $\vec{u} \times \vec{v} = (3, 0, 0) \times (1, 2, 0) = (0, 0, 6)$, so the exact area is $\|\vec{u} \times \vec{v}\| = \boxed{6}$ – close to the estimate.

Exercise 6.4

$A = (1, -2, 0)$, $B = (4, -1, 3)$, $C = (6, 3, 3)$, $D = (3, 2, 0)$.

(a) $\overrightarrow{AB} = (3, 1, 3)$ and $\overrightarrow{DC} = C - D = (3, 1, 3)$. Since $\overrightarrow{AB} = \overrightarrow{DC}$, $ABCD$ **is** a parallelogram.

(b) Using adjacent sides $\overrightarrow{AB} = (3, 1, 3)$ and $\overrightarrow{AD} = (2, 4, 0)$:

$$\overrightarrow{AB} \times \overrightarrow{AD} = (1 \cdot 0 - 3 \cdot 4,\ 3 \cdot 2 - 3 \cdot 0,\ 3 \cdot 4 - 1 \cdot 2) = (-12, 6, 10).$$

Area $= \|(-12, 6, 10)\| = \sqrt{144 + 36 + 100} = \sqrt{280} = \boxed{2\sqrt{70} \approx 16.7}$.

Exercise 6.5

$A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$.

(a) $\overrightarrow{AB} = (-1, 1, 0)$, $\overrightarrow{AC} = (-1, 0, 1)$.

(b) $\overrightarrow{AB} \times \overrightarrow{AC} = (1 \cdot 1 - 0 \cdot 0,\ 0 \cdot (-1) - (-1) \cdot 1,\ (-1) \cdot 0 - 1 \cdot (-1)) = \boxed{(1, 1, 1)}$.

(c) This vector is **normal to the plane** through A, B, C (the plane $x + y + z = 1$). Its norm is $\sqrt{3}$, so the unit normal is $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

(d) Area $= \dfrac{1}{2}\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\| = \boxed{\dfrac{\sqrt{3}}{2}}$.

Exercise 6.6

$\vec{r} = (0.3, 0, 0)$ m.

(a) $\vec{\tau} = \vec{r} \times \vec{F} = (0.3, 0, 0) \times (12, 16, 0) = (0 \cdot 0 - 0 \cdot 16,\ 0 \cdot 12 - 0.3 \cdot 0,\ 0.3 \cdot 16 - 0 \cdot 12) = \boxed{(0, 0, 4.8)}$ N·m, magnitude $\boxed{4.8}$ N·m.

(b) $\sin\theta = \dfrac{\|\vec{\tau}\|}{\|\vec{r}\|\left\|\vec{F}\right\|} = \dfrac{4.8}{0.3 \cdot 20} = \dfrac{4.8}{6} = 0.8 \implies \boxed{\theta \approx 53.1^\circ}$.

This agrees with the geometry: $\vec{F} = (12, 16, 0)$ makes an angle $\arctan(16/12) \approx 53.1^\circ$ with the handle (the x-axis) – a 3–4–5 triangle scaled by 4.

(c) Maximum torque occurs at $\theta = 90^\circ$ (force perpendicular to the handle):

$$\tau_{\max} = \|\vec{r}\|\left\|\vec{F}\right\| = 0.3 \times 20 = \boxed{6\ \mathrm{N} \cdot \mathrm{m}}.$$

(d) $\vec{\tau} = (0.3, 0, 0) \times (20, 0, 0) = \boxed{\vec{0}}$. Pushing straight along the handle has no component perpendicular to it, so there is nothing to create a turning effect – all the force goes into stretching/compressing the wrench rather than rotating it.

Collinearity and Coplanarity

Exercise 7.1

(a) $\vec{u} = (3, -6, 9) = -3(-1, 2, -3) = -3\vec{v}$, so $\vec{u} \times \vec{v} = \boxed{\vec{0}}$: the vectors **are collinear**.

(b) $\vec{u} \times \vec{v} = (1, 2, 3) \times (2, 4, 5) = (2 \cdot 5 - 3 \cdot 4,\ 3 \cdot 2 - 1 \cdot 5,\ 1 \cdot 4 - 2 \cdot 2) = (-2, 1, 0) \neq \vec{0}$: **not collinear**. (By inspection: $2/1 = 2$ but $5/3 \neq 2$, so $(2, 4, 5)$ is not a scalar multiple of $(1, 2, 3)$.)

Exercise 7.2

$P = (1, 1)$, $Q = (3, 2)$, $R = (5, 3)$, $S = (4, 4)$ (embedded in $\mathbb{R}^3$ with $z = 0$).

(a) By inspection, P, Q, R appear to rise at a constant rate and look **collinear**.

(b) $\overrightarrow{PQ} = (2, 1, 0)$, $\overrightarrow{PR} = (4, 2, 0) = 2\,\overrightarrow{PQ}$, so $\overrightarrow{PQ} \times \overrightarrow{PR} = \vec{0}$ – **confirmed collinear**.

(c) $\overrightarrow{PS} = (3, 3, 0)$. $\overrightarrow{PQ} \times \overrightarrow{PS} = (1 \cdot 0 - 0 \cdot 3,\ 0 \cdot 3 - 2 \cdot 0,\ 2 \cdot 3 - 1 \cdot 3) = (0, 0, 3) \neq \vec{0}$, so S is $\boxed{\textbf{not} \text{ on the line through } P, Q, R}$.

Scalar Triple Product

Exercise 8.1

(a) These three vectors are mutually perpendicular and axis-aligned, so the parallelepiped is just a rectangular box: predicted volume $= 2 \times 3 \times 5 = 30$. Formally,

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{vmatrix} = \boxed{30}.$$

(b) $\vec{v} \times \vec{w} = (0, 1, 3) \times (2, 0, 1) = (1 \cdot 1 - 3 \cdot 0,\ 3 \cdot 2 - 0 \cdot 1,\ 0 \cdot 0 - 1 \cdot 2) = (1, 6, -2)$.

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = (1)(1) + (2)(6) + (0)(-2) = \boxed{13}.$$

Exercise 8.2

(a) $\vec{v} \times \vec{w} = (0, 1, 1) \times (1, 1, 2) = (1 \cdot 2 - 1 \cdot 1,\ 1 \cdot 1 - 0 \cdot 2,\ 0 \cdot 1 - 1 \cdot 1) = (1, 1, -1)$.

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = 1 + 0 - 1 = \boxed{0} \quad \Longrightarrow \quad \textbf{coplanar}.$$

Geometrically, $\vec{w} = \vec{u} + \vec{v}$ here, since $(1, 1, 2) = (1, 0, 1) + (0, 1, 1)$, so $\vec{w}$ already lies in the plane spanned by $\vec{u}$ and $\vec{v}$.

(b) $\vec{v} \times \vec{w} = (0, 1, 1) \times (1, 0, 1) = (1 \cdot 1 - 1 \cdot 0,\ 1 \cdot 1 - 0 \cdot 1,\ 0 \cdot 0 - 1 \cdot 1) = (1, 1, -1)$.

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = 1 + 1 - 0 = \boxed{2} \neq 0 \quad \Longrightarrow \quad \textbf{not coplanar}.$$

Exercise 8.3

$\vec{u} = (1, 0, 1)$, $\vec{v} = (0, 1, 0)$, $\vec{w} = (1, 1, 0)$.

(a) $\vec{u} \cdot \vec{v} = 0$: $\boxed{\text{orthogonal}}$.

(b) $\vec{u} \times \vec{v} = (0 \cdot 0 - 1 \cdot 1,\ 1 \cdot 0 - 1 \cdot 0,\ 1 \cdot 1 - 0 \cdot 0) = \boxed{(-1, 0, 1)}$.

(c)

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 1(1 \cdot 0 - 0 \cdot 1) - 0(0 \cdot 0 - 0 \cdot 1) + 1(0 \cdot 1 - 1 \cdot 1) = 0 - 0 - 1 = \boxed{-1}.$$

(d) Since the triple product is $-1 \neq 0$, $\vec{u}, \vec{v}, \vec{w}$ are $\boxed{\textbf{not coplanar}}$.

(e) $\|\vec{u} \times \vec{v}\| = \sqrt{1 + 0 + 1} = \sqrt{2}$, so $\sin\theta = \dfrac{\sqrt{2}}{\|\vec{u}\|\,\|\vec{v}\|} = \dfrac{\sqrt{2}}{\sqrt{2} \cdot 1} = 1 \implies \boxed{\theta = 90^\circ}$.

(f) This matches (a) directly: $\vec{u} \cdot \vec{v} = 0 \;\Rightarrow\; \theta = 90^\circ$. **Both methods agree.**

Exercise 8.4

Throughout, write $D := \vec{u} \cdot (\vec{v} \times \vec{w})$, and recall the two key facts about the scalar triple product: it is unchanged under a *cyclic* permutation of its three vectors, and it changes sign under a *transposition* (swapping any two of them). In particular

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{v} \cdot (\vec{w} \times \vec{u}) = \vec{w} \cdot (\vec{u} \times \vec{v}) = D, \qquad \vec{w} \cdot (\vec{v} \times \vec{u}) = -D.$$

(a) Since $\vec{u} \times \vec{u} = \vec{v} \times \vec{v} = \vec{0}$ and $\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$,

$$(\vec{u} + \vec{v}) \times (\vec{u} - \vec{v}) = -\vec{u} \times \vec{v} + \vec{v} \times \vec{u} = -2(\vec{u} \times \vec{v}).$$

Dotting with $\vec{u}$ gives 0 (a vector is always orthogonal to $\vec{u} \times \vec{v}$'s partner $\vec{u}$). Also $-\vec{w} \cdot (\vec{v} \times \vec{u}) = +\vec{w} \cdot (\vec{u} \times \vec{v}) = D$. So the whole expression is

$$0 + 2D + D = \boxed{3D = 3\,\vec{w} \cdot (\vec{u} \times \vec{v})}.$$

(b) By the BAC-CAB rule, $\vec{a} \times (\vec{a} \times \vec{b}) = (\vec{a} \cdot \vec{b})\vec{a} - \|\vec{a}\|^2\,\vec{b}$. Adding $\|\vec{a}\|^2\,\vec{b}$ cancels the second term, leaving

$$\boxed{(\vec{a} \cdot \vec{b})\,\vec{a}}.$$

(c) Using the identities above: $\vec{u} \cdot (\vec{v} \times \vec{w}) = D$, $\vec{v} \cdot (\vec{w} \times \vec{u}) = D$, $\vec{w} \cdot (\vec{u} \times \vec{v}) = D$, and $\vec{w} \cdot (\vec{v} \times \vec{u}) = -D$. So

$$D - D + D - 3(-D) = \boxed{4D = 4\,\vec{u} \cdot (\vec{v} \times \vec{w})}.$$

(d) As in part (a), $[(\vec{u} + \vec{v}) \times (\vec{u} - \vec{v})] \cdot \vec{w} = -2(\vec{u} \times \vec{v}) \cdot \vec{w} = -2D$. Next, $2\vec{w} \cdot (\vec{v} \times \vec{u}) = -2D$. Then

$$\vec{u} \cdot [\vec{u} \times (\vec{v} + \vec{w})] = \vec{u} \cdot (\vec{u} \times \vec{v}) + \vec{u} \cdot (\vec{u} \times \vec{w}) = 0 + 0 = 0,$$

so that term contributes 0. Finally $\dfrac{\|\vec{u}\|^2}{3} - \vec{u} \cdot \vec{u} = \dfrac{\|\vec{u}\|^2}{3} - \|\vec{u}\|^2 = -\dfrac{2}{3}\,\|\vec{u}\|^2$. Adding everything:

$$\boxed{-4D - \tfrac{2}{3}\,\|\vec{u}\|^2}, \qquad D = \vec{u} \cdot (\vec{v} \times \vec{w}).$$

(e) By BAC-CAB, $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$. Dotting with $\vec{u}$:

$$(\vec{u} \cdot \vec{w})(\vec{u} \cdot \vec{v}) - (\vec{u} \cdot \vec{v})(\vec{u} \cdot \vec{w}) = 0.$$

Exactly the same cancellation occurs for the other two (cyclically symmetric) terms, so the entire sum is $\boxed{0}$.

(f) This is precisely the vector **Jacobi identity**. Expanding each term with BAC-CAB,

$$\begin{aligned}
\vec{u} \times (\vec{v} \times \vec{w}) &= (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}, \\
\vec{v} \times (\vec{w} \times \vec{u}) &= (\vec{v} \cdot \vec{u})\vec{w} - (\vec{v} \cdot \vec{w})\vec{u}, \\
\vec{w} \times (\vec{u} \times \vec{v}) &= (\vec{w} \cdot \vec{v})\vec{u} - (\vec{w} \cdot \vec{u})\vec{v},
\end{aligned}$$

and every pair of terms cancels when summed ($\vec{v}$-terms: $(\vec{u} \cdot \vec{w}) - (\vec{w} \cdot \vec{u}) = 0$, and similarly for $\vec{u}$- and $\vec{w}$-terms). So the sum is $\boxed{\vec{0}}$.

(g) Using the sign rules: $\vec{v} \cdot (\vec{u} \times \vec{w}) = -D$, $\vec{w} \cdot (\vec{v} \times \vec{u}) = -D$, $\vec{v} \cdot (\vec{w} \times \vec{u}) = D$, $\vec{w} \cdot (\vec{u} \times \vec{v}) = D$. So

$$2D + (-D) - (-D) - 4(D) + 3(D) = 2D - D + D - 4D + 3D = \boxed{D = \vec{u} \cdot (\vec{v} \times \vec{w})}.$$

(h) Similarly, with $D = \vec{a} \cdot (\vec{b} \times \vec{c})$: $\vec{b} \cdot (\vec{a} \times \vec{c}) = -D$, $\vec{c} \cdot (\vec{b} \times \vec{a}) = -D$, $\vec{a} \cdot (\vec{c} \times \vec{b}) = -D$. So

$$2D + 3(-D) - (-D) + (-D) = 2D - 3D + D - D = \boxed{-D = -\,\vec{a} \cdot (\vec{b} \times \vec{c})}.$$