Chapter 2 Summary

Norm, Distance, Dot Product, Projection, and Cross Product

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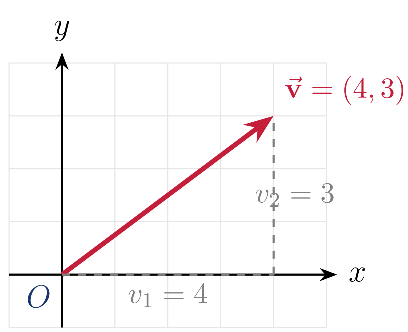
1 Norm of a Vector

Norm (length) of a vector

For $\vec{\mathbf{v}} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$,

$$\|\vec{\mathbf{v}}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

The formula is just Pythagoras applied once (in $\mathbb{R}^2$) or twice (in $\mathbb{R}^3$): the floor diagonal $\sqrt{v_1^2 + v_2^2}$ becomes one leg of a second right triangle with the height v_3. For $n \geq 4$ there is no picture, but the same pattern holds.



Key properties. $\|\vec{\mathbf{v}}\| \geq 0$ always, and $\|\vec{\mathbf{v}}\| = 0 \iff \vec{\mathbf{v}} = \vec{\mathbf{0}}$. Scaling: $\|c\vec{\mathbf{v}}\| = |c|\,\|\vec{\mathbf{v}}\|$ (the absolute value matters, negating a vector flips direction, not length). Triangle inequality: $\|\vec{\mathbf{u}} + \vec{\mathbf{v}}\| \leq \|\vec{\mathbf{u}}\| + \|\vec{\mathbf{v}}\|$, with equality only when $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ point in the same direction.

Worked example

Let $\vec{\mathbf{v}} = (1, -2, 2)$. Then $\|\vec{\mathbf{v}}\| = \sqrt{1 + 4 + 4} = \sqrt{9} = 3$. Now check the scaling property with $c = -3$: $\|-3\vec{\mathbf{v}}\| = |-3| \cdot 3 = 9$, *not* -9 (a negative length makes no sense).

2 Distance Between Two Points

Distance

For points $A, B \in \mathbb{R}^n$, the distance from A to B is the length of the vector $\overrightarrow{AB} = B - A$:

$$d(A,B) = \|\overrightarrow{AB}\| = \|B - A\| = \sqrt{(b_1 - a_1)^2 + \cdots + (b_n - a_n)^2}.$$

Distance is symmetric ($d(A,B) = d(B,A)$), $d(A,A) = 0$, and it satisfies the triangle inequality $d(A,C) \le d(A,B) + d(B,C)$ (all inherited directly from the norm).

Worked example

$A = (1,1)$, $B = (5,4)$: $\overrightarrow{AB} = (4,3)$, so $d(A,B) = \sqrt{16+9} = 5$.

3 Unit Vectors and Normalization

Unit vector & normalization

A vector $\vec{\mathbf{u}}$ is a **unit vector** if $\|\vec{\mathbf{u}}\| = 1$. For any nonzero $\vec{\mathbf{v}}$, the unit vector in the direction of $\vec{\mathbf{v}}$ is

$$\hat{\vec{\mathbf{v}}} = \frac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|}.$$

Dividing by the norm rescales the vector to length exactly 1 while keeping the same direction, useful whenever only direction matters (compass bearings, surface normals). Normalization is undefined for $\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

Worked example

$\vec{\mathbf{v}} = (4,3)$: $\|\vec{\mathbf{v}}\| = 5$, so $\hat{\vec{\mathbf{v}}} = \left(\frac{4}{5}, \frac{3}{5}\right) = (0.8, 0.6)$, and indeed $\|\hat{\vec{\mathbf{v}}}\| = \sqrt{0.64 + 0.36} = 1$.

4 The Dot Product

Dot product

For $\vec{\mathbf{u}} = (u_1, \ldots, u_n)$, $\vec{\mathbf{v}} = (v_1, \ldots, v_n)$:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Equivalently, if θ is the angle between them,

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \|\vec{\mathbf{u}}\| \, \|\vec{\mathbf{v}}\| \cos\theta.$$

The dot product is a *scalar* that measures alignment: positive when the vectors lean toward each other ($\theta < 90°$), zero when perpendicular, negative when they lean apart.

Properties of the dot product

For all $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}} \in \mathbb{R}^n$ and all scalars $c \in \mathbb{R}$:

1. **Commutativity:** $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{v}} \cdot \vec{\mathbf{u}}$
2. **Distributivity:** $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} + \vec{\mathbf{u}} \cdot \vec{\mathbf{w}}$
3. **Scalar compatibility:** $(c\vec{\mathbf{u}}) \cdot \vec{\mathbf{v}} = c(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}) = \vec{\mathbf{u}} \cdot (c\vec{\mathbf{v}})$
4. **Positive definiteness:** $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} \geq 0$, and $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = 0 \iff \vec{\mathbf{v}} = \vec{\mathbf{0}}$

Property 4 is exactly why $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = \|\vec{\mathbf{v}}\|^2$ and $\|\vec{\mathbf{v}}\| = \sqrt{\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}}$ (the norm is a special case of the dot product). Properties 1–2 are what let you expand $(\vec{\mathbf{u}} + \vec{\mathbf{v}}) \cdot (\vec{\mathbf{u}} + \vec{\mathbf{v}})$ term by term, exactly like $(a+b)^2$ in ordinary algebra.

Be careful

$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$ is a number, not a vector, and it is *not* componentwise multiplication: $(1,2) \cdot (3,4) = 1(3) + 2(4) = 11$, not $(3,8)$.

Worked example

$\vec{\mathbf{u}} = (1,0)$, $\vec{\mathbf{v}} = (1,1)$: $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 1$, $\|\vec{\mathbf{u}}\| = 1$, $\|\vec{\mathbf{v}}\| = \sqrt{2}$, so $\cos\theta = \frac{1}{\sqrt{2}}$ and $\theta = 45°$.

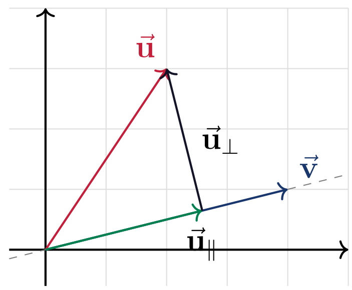
5 Projection and Applications

Projection of $\vec{\mathbf{u}}$ onto $\vec{\mathbf{v}}$

For $\vec{\mathbf{v}} \neq \vec{\mathbf{0}}$:

$$\mathrm{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{u}}) = \frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|^2}\,\vec{\mathbf{v}}.$$

This splits $\vec{\mathbf{u}}$ into a piece along $\vec{\mathbf{v}}$ and a piece orthogonal to it. The scalar $\frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|^2}$ tells you how far along $\vec{\mathbf{v}}$ to go; multiplying by $\vec{\mathbf{v}}$ gives the direction. Physically, this is how **work** is defined: $W = \vec{\mathbf{F}} \cdot \vec{\mathbf{d}}$ (only the component of force along the direction of motion contributes).



Worked example

$\vec{\mathbf{u}} = (\frac{3}{2}, 3)$, $\vec{\mathbf{v}} = (3,2)$: $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \frac{21}{2}$, $\|\vec{\mathbf{v}}\|^2 = 13$, so $\mathrm{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{u}}) = \frac{21}{26}(3,2)$.

6 The Cross Product

Cross product in $\mathbb{R}^3$

For $\vec{\mathbf{u}} = (u_1, u_2, u_3)$, $\vec{\mathbf{v}} = (v_1, v_2, v_3)$:

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1).$$

Unlike the dot product, the cross product returns a *vector*, specifically, one perpendicular to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ (right-hand rule for orientation).

Properties of the cross product

For all $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}} \in \mathbb{R}^3$ and a scalar c:

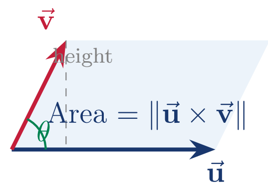
1. **Anticommutativity:** $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = -(\vec{\mathbf{v}} \times \vec{\mathbf{u}})$ — order matters
2. **Scalar compatibility:** $(c\vec{\mathbf{u}}) \times \vec{\mathbf{v}} = c(\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = \vec{\mathbf{u}} \times (c\vec{\mathbf{v}})$
3. **Distributivity:** $\vec{\mathbf{u}} \times (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \times \vec{\mathbf{v}} + \vec{\mathbf{u}} \times \vec{\mathbf{w}}$
4. **Self cross product:** $\vec{\mathbf{u}} \times \vec{\mathbf{u}} = \vec{\mathbf{0}}$

Magnitude: the area of the parallelogram

If θ is the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$:

$$\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\| = \|\vec{\mathbf{u}}\|\,\|\vec{\mathbf{v}}\| \sin\theta.$$

This is exactly the **area of the parallelogram** spanned by $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ (base $\times$ height). For the triangle with sides $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, the area is half that: $\text{Area}_\triangle = \frac{1}{2}\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|$.



Collinearity test

$\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are collinear $\iff$ $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \vec{\mathbf{0}}$ (since $\sin\theta = 0$ exactly when $\theta = 0$ or π).

Worked example

$\vec{\mathbf{u}} = (1, 2, 3)$, $\vec{\mathbf{v}} = (4, 5, 6)$: $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (2\cdot6 - 3\cdot5,\ 3\cdot4 - 1\cdot6,\ 1\cdot5 - 2\cdot4) = (-3, 6, -3)$. Check: $\vec{\mathbf{u}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = -3 + 12 - 9 = 0$ ✓, confirming perpendicularity. The area of the parallelogram is $\|(-3, 6, -3)\| = \sqrt{54} = 3\sqrt{6}$.

7 The Scalar Triple Product

Scalar triple product

For $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}} \in \mathbb{R}^3$:

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

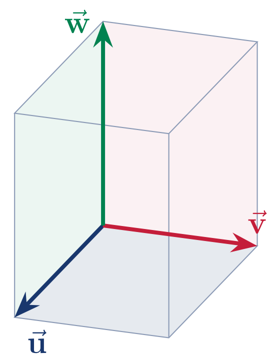
While the dot product turns two vectors into a scalar (alignment) and the cross product turns two vectors into a vector (area, perpendicularity), the scalar triple product turns *three* vectors into a scalar.

Geometric meaning: volume of a parallelepiped

$\vec{\mathbf{v}} \times \vec{\mathbf{w}}$ gives the area of the base spanned by $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$; dotting with $\vec{\mathbf{u}}$ then extracts the height of $\vec{\mathbf{u}}$ above that base. So

$$\text{Volume} = |\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})|$$

is the volume of the parallelepiped generated by $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$.



Properties of the scalar triple product

For any $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}} \in \mathbb{R}^3$:

1. **Cyclic permutation** does not change the value:

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = \vec{\mathbf{v}} \cdot (\vec{\mathbf{w}} \times \vec{\mathbf{u}}) = \vec{\mathbf{w}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}})$$

2. **Swapping any two vectors reverses the sign**, e.g.

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = -\vec{\mathbf{u}} \cdot (\vec{\mathbf{w}} \times \vec{\mathbf{v}}) = -\vec{\mathbf{v}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{w}})$$

3. The **volume** $|\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})|$ is unchanged by either operation, since a permutation or sign flip never changes an absolute value.

Coplanarity test

$\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}$ are coplanar $\iff$ $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = 0$, the parallelepiped flattens to zero volume.

Worked example

$\vec{\mathbf{u}} = (3, 0, 0)$, $\vec{\mathbf{v}} = (0, 2, 0)$, $\vec{\mathbf{w}} = (0, 0, 4)$: $\vec{\mathbf{v}} \times \vec{\mathbf{w}} = (8, 0, 0)$, so $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = 24$, matching the box volume $3 \times 2 \times 4 = 24$ directly.