Chapter 3 - Exercises

Lines in the Plane and in Space

Equations, Orthogonal Projection, Distances, Relative Position, and Skew Lines

Collège Sainte-Anne | Fall 2026

Defining a Line

Reference Formulas

Line through P_0 with direction $\vec{\mathbf{d}} = (a, b)$: vector form $(x, y) = P_0 + t\vec{\mathbf{d}}$; parametric form $x = x_0 + ta,\ y = y_0 + tb$; Cartesian form $b(x - x_0) - a(y - y_0) = 0$, i.e. $Ax + By + C = 0$ with normal vector $\vec{\mathbf{n}} = (A, B) = (b, -a)$. From $Ax + By + C = 0$ a direction vector is $(-B, A)$. In $\mathbb{R}^3$, a single linear Cartesian equation describes a plane, not a line. For now, use vector or parametric form for a line; a Cartesian description of a line in $\mathbb{R}^3$ requires two independent plane equations (and, when appropriate, one may also use symmetric equations).

1. ⋆ **Two points.** Let $A(2, -1)$ and $B(5, 3)$.
   (a) Find a direction vector $\vec{\mathbf{d}}$.
   (b) Write the vector, parametric and Cartesian equations of line AB.
   (c) *Check:* verify that both A and B satisfy your Cartesian equation.

2. ⋆ **From Cartesian to vector.** A line has equation $3x - 2y + 6 = 0$.
   (a) Find a point on it and a direction vector.
   (b) Write its vector and parametric equations.
   (c) *Check:* show your direction vector is perpendicular to $\vec{\mathbf{n}} = (3, -2)$.

3. ⋆ **A line in space.** Let $P_0(1, -2, 3)$ and $\vec{\mathbf{d}} = (2, 0, -1)$.
   (a) Write the vector and parametric equations of the line through P_0 with direction $\vec{\mathbf{d}}$.
   (b) Find the point at $t = 3$.
   (c) Do $(7, -2, 0)$ and $(3, -1, 2)$ lie on the line?

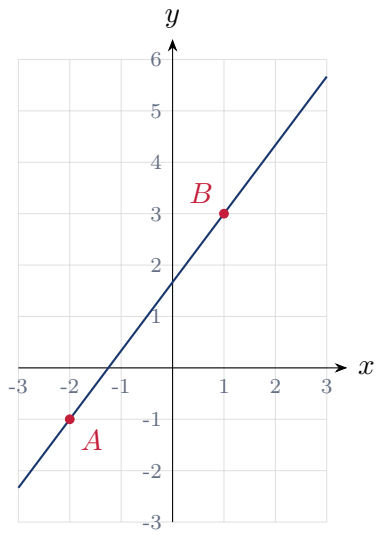
4. ⋆ **Vertical and horizontal lines.**

   (a) Find the Cartesian equation of the line through $(2, 1)$ with direction $(0, 3)$, and of the line through $(2, 1)$ with direction $(5, 0)$.

   (b) Try to write each in the "fraction form" $\frac{x-x_0}{a} = \frac{y-y_0}{b}$. What goes wrong?

   (c) Why does $b(x - x_0) - a(y - y_0) = 0$ not have this problem?

5. ⋆⋆ **Read from the graph.** Two points A and B of a line are marked.

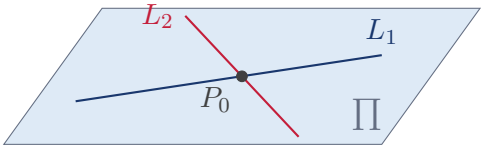


   (a) Read the coordinates of A and B.

   (b) Find the vector, parametric and Cartesian equations of the line.

   (c) Use your Cartesian equation to *predict* where the line crosses the y-axis, then check on the graph.

6. ⋆⋆ EXT. **Why one equation is not enough in $\mathbb{R}^3$.** The shaded region below is a piece of the plane $\Pi : x + y + z = 3$. The point $P_0 = (1, 1, 1)$ lies in Π, and so do the two lines L_1 (direction $(1, -1, 0)$) and L_2 (direction $(1, 1, -2)$) through P_0.



   (a) Substitute the parametric equations of L_1 and of L_2 into $x + y + z = 3$ and confirm they satisfy it for every t.

   (b) How many different lines lie in Π and pass through P_0?

   (c) What does this tell you about describing a line in $\mathbb{R}^3$ with a single equation?

7. ⋆⋆ EXT. **Find the error.** A student writes: "the line through $P_0(1, 2)$ with direction $\vec{\mathbf{d}} = (3, 4)$ has Cartesian equation $3(x - 1) = 4(y - 2)$."

(a) Give a one-line test, using a point you *know* is on the line, that shows the equation is wrong.

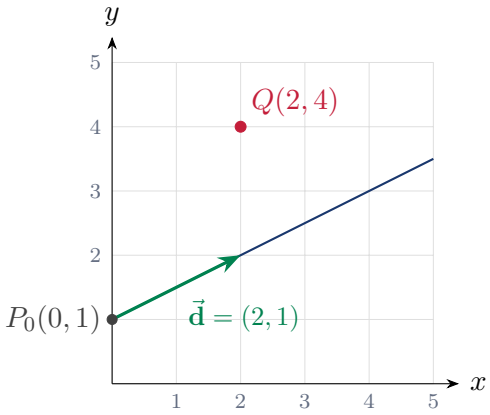
(b) Correct it, and explain the mistake in terms of the normal vector.

Orthogonal Projection of a Point on a Line

Reference Formulas

With $\vec{\mathbf{w}} = Q - P_0$: $\quad t = \dfrac{\vec{\mathbf{w}} \cdot \vec{\mathbf{d}}}{\vec{\mathbf{d}} \cdot \vec{\mathbf{d}}}, \quad H = P_0 + t\,\vec{\mathbf{d}}$. The vector $\overrightarrow{HQ}$ is perpendicular to $\vec{\mathbf{d}}$. Same recipe in $\mathbb{R}^2$ and $\mathbb{R}^3$.

8. ⋆ **Estimate, then compute.** The line L passes through $P_0(0,1)$ with direction $\vec{\mathbf{d}} = (2,1)$.



(a) By eye, estimate the foot H of the perpendicular from Q to L, and the distance QH.

(b) Compute H exactly.

(c) *Check:* verify $\overrightarrow{HQ} \cdot \vec{\mathbf{d}} = 0$, then compare with your estimate.

9. ⋆ **Projection in the plane.** Project $Q(6,2)$ onto the line through the origin with direction $(1,3)$, and verify perpendicularity.

10. ⋆⋆ **Projection in space.** Project $Q(2,-1,5)$ onto the line through $P_0(1,1,1)$ with direction $\vec{\mathbf{d}} = (1,1,-1)$, and verify perpendicularity.

11. ⋆⋆ **Closest point.** Find the point of the line $(x,y) = (2,1) + t(1,2)$ closest to the origin, and the distance from the origin to the line. (Here $Q = (0,0)$.)

12. ⋆⋆ EXT. **A point already on the line.** Show that if Q lies on L, its orthogonal projection onto L is Q itself. *Hint:* write $Q = P_0 + k\vec{\mathbf{d}}$.

13. ⋆⋆ EXT. **Does the choice of $\vec{\mathbf{d}}$ matter?** Replace $\vec{\mathbf{d}}$ by $\vec{\mathbf{d}}' = k\vec{\mathbf{d}}$ $(k \neq 0)$. Show the projection formula produces the same point H. Why should you have expected this geometrically?

Distance from a Point to a Line

Reference Formulas

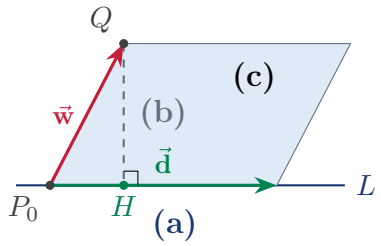
$d(Q, L) = \|Q - H\|$. In $\mathbb{R}^2$ with $L : Ax + By + C = 0$: $\dfrac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$. In $\mathbb{R}^3$: $\dfrac{\|\vec{\mathbf{d}} \times \vec{\mathbf{w}}\|}{\|\vec{\mathbf{d}}\|}$ with $\vec{\mathbf{w}} = Q - P_0$. There is no analogous point-to-line shortcut based on a single linear Cartesian equation, because one such equation describes a plane rather than a line.

14. ⋆ **Cartesian shortcut.** Find the distance from $Q(5, 5)$ to the line $3x + 4y - 12 = 0$.

15. ⋆⋆ **Two methods.** Let L pass through $P_0(1, 1)$ with direction $(2, 1)$, and $Q = (6, 4)$.
    (a) Find $d(Q, L)$ by projection.
    (b) Find the Cartesian equation of L and recompute with the shortcut.
    (c) Which method was quicker here? Which would you have to use in $\mathbb{R}^3$?

16. ⋆⋆ **Projection in space.** Let L pass through $P_0(1, 0, 1)$ with $\vec{\mathbf{d}} = (2, 1, 2)$, and $Q = (4, 0, 4)$.
    (a) Find the orthogonal projection H of Q on L.
    (b) Find $d(Q, L)$.
    (c) *Check:* verify $\overrightarrow{HQ} \cdot \vec{\mathbf{d}} = 0$.

17. ⋆⋆ **Cross-product shortcut.** Same L and Q as the previous problem.
    (a) Compute $d(Q, L) = \|\vec{\mathbf{d}} \times \vec{\mathbf{w}}\| / \|\vec{\mathbf{d}}\|$ and compare with your earlier answer.
    (b) In the figure, each of (a), (b), (c) is one of: $\|\vec{\mathbf{d}}\|$, $d(Q, L)$, $\|\vec{\mathbf{d}} \times \vec{\mathbf{w}}\|$. Identify them.
    (c) Use the figure to explain why the formula in (a) holds.



18. ⋆⋆ **Applied.** The axis of a laser beam is modeled by the line through the origin with direction $(1, 2, 2)$. A sensor sits at $(9, 0, 0)$.
    (a) Find the minimum distance between the sensor and the beam.
    (b) At which point of the beam is the sensor closest?

19. ⋆⋆ EXT. **Find the error.** To find the distance from $Q(1, 2, 3)$ to the line through the origin with direction $(1, 1, 0)$, a student writes: "the line is $x - y = 0$, so $d = \frac{|1 \cdot 1 - 1 \cdot 2|}{\sqrt{2}} = \frac{\sqrt{2}}{2}$."

(a) What does $x - y = 0$ actually describe in $\mathbb{R}^3$, and what distance did the student compute?

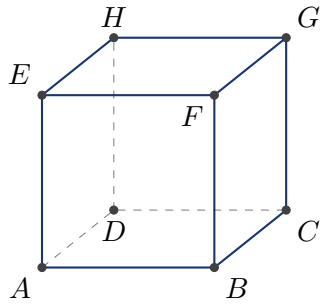
(b) Compute the correct distance.

Relative Position of Two Lines

Reference Formulas

(1) Directions parallel? ($\vec{\mathbf{v}} = k\vec{\mathbf{u}}$, or $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \vec{\mathbf{0}}$ in $\mathbb{R}^3$.) (2) If parallel: is $P_2 \in L_1$? Yes $\Rightarrow$ equal; no $\Rightarrow$ distinct. (3) If not parallel: in $\mathbb{R}^2$ solve the 2×2 system for (t, s); in $\mathbb{R}^3$ solve *two independent* coordinate equations, then **test the third**: holds $\Rightarrow$ intersect, fails $\Rightarrow$ skew.

20. ⋆⋆ **Lines on a cube.** Consider the unit cube with $A(0,0,0)$, $B(1,0,0)$, $C(1,1,0)$, $D(0,1,0)$, $E(0,0,1)$, $F(1,0,1)$, $G(1,1,1)$, $H(0,1,1)$.



For each pair of lines below, first *predict* from the picture: equal, parallel and distinct, intersecting, or skew. Then *confirm* with the algebraic test.

(a) (AB) and (EF) (b) (AB) and (BC) (c) (AB) and (CG)

(d) (AG) and (BH) (two space diagonals) (e) (AG) and (BC)

Warning: in a flat drawing, skew lines can look as though they cross. Only the algebra is reliable.

21. ⋆ **Parallel: equal or distinct?**

(a) In $\mathbb{R}^2$: L_1 through $(1,2)$ with direction $(2,-1)$, and L_2 through $(3,0)$ with direction $(-4,2)$.

(b) In $\mathbb{R}^3$: L_1 through $(1,0,0)$ with $\vec{\mathbf{u}} = (1,1,0)$, and L_2 through $(0,1,1)$ with $\vec{\mathbf{v}} = (2,2,0)$.

22. ⋆ **Intersection in the plane.** Find the intersection of $L_1 : (0,1) + t(1,1)$ and $L_2 : (4,-3) + s(-1,2)$. Check your answer in *both* equations.

23. ⋆⋆ **Skew test.** L_1 passes through the origin with $\vec{\mathbf{u}} = (1,0,1)$; L_2 passes through $(1,1,0)$ with $\vec{\mathbf{v}} = (0,1,1)$. Intersect or skew? *Hint:* solve the x- and y-equations, then test z.

24. ⋆⋆ **Intersection in space.** L_1 passes through the origin with $\vec{\mathbf{u}} = (1, 1, 1)$; L_2 passes through $(3, -1, 2)$ with $\vec{\mathbf{v}} = (-2, 2, -1)$.
    (a) Intersect or skew? If they intersect, find the point.
    (b) Redo it solving the y- and z-equations first and testing x. Do you reach the same conclusion? Why must you?

25. ⋆⋆ EXT. **Find the error.** For $L_1 : (t, t, 2t)$ and $L_2 : (1 + s, 2s, -s)$, a student solves the x- and y-equations, finds $t = 2$, $s = 1$, and concludes "the lines intersect at $(2, 2, 4)$." What did the student forget? What is the true relationship?

26. ⋆⋆⋆ EXT. **A parameter.** Let $L_1 : (t, t, t)$ and $L_2 : (1,\ s,\ k + s)$. For which values of k do the lines intersect? What is their relationship for all other values of k?

Distance Between Two Lines

Reference Formulas

Parallel: $d(L_1, L_2) = d(P_2, L_1)$. Skew, normal equations: $(P_1 - P_2)\cdot\vec{\mathbf{u}} + t(\vec{\mathbf{u}}\cdot\vec{\mathbf{u}}) - s(\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}) = 0$ and $(P_1 - P_2)\cdot\vec{\mathbf{v}} + t(\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}) - s(\vec{\mathbf{v}}\cdot\vec{\mathbf{v}}) = 0$; then $d = \|H_1 - H_2\|$. Skew, shortcut: $\dfrac{|(P_2 - P_1)\cdot(\vec{\mathbf{u}} \times \vec{\mathbf{v}})|}{\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|}$. Intersecting or equal: $d = 0$.

27. ⋆ **Parallel lines in the plane.** L_1 passes through $(0, 0)$ with direction $(1, 2)$; L_2 passes through $(5, 0)$ with the same direction.
    (a) Find $d(L_1, L_2)$ using projection.
    (b) Confirm using the $\mathbb{R}^2$ point-to-line shortcut.

28. ⋆⋆ **Distances on the cube.** Same unit cube as Problem 20.
    (a) Guess from the picture, then compute: $d\big((AB), (EF)\big)$ and $d\big((AB), (CG)\big)$.
    (b) Compute $d\big((AG), (BC)\big)$. Is the answer what you expected?

29. ⋆⋆ **A genuine skew pair.** L_1 passes through $P_1(0, 0, 0)$ with $\vec{\mathbf{u}} = (1, 1, 0)$; L_2 passes through $P_2(1, 0, 1)$ with $\vec{\mathbf{v}} = (0, 1, 1)$.
    (a) Show the lines are skew.
    (b) Write the two normal equations and solve them by substitution.
    (c) Find H_1, H_2 and $d(L_1, L_2)$.
    (d) *Check:* verify $\overrightarrow{H_1H_2}$ is perpendicular to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.
    (e) Confirm d with the cross-product shortcut.

30. ⋆⋆⋆ EXT. **The one-variable route.** Same lines as Problem 29. Let $P_t = P_1 + t\vec{\mathbf{u}}$ and let Q_t be its orthogonal projection onto L_2.

(a) Find $s(t)$ such that $Q_t = P_2 + s(t)\vec{\mathbf{v}}$. What kind of function of t is it?

(b) Express $f(t)^2 = \|P_t - Q_t\|^2$ as a polynomial in t.

(c) Minimize it. Compare t^* and $\min f$ with Problem 29.

(d) Show that $(P_t - Q_t) \cdot \vec{\mathbf{v}} = 0$ for *every* t, and that $(P_{t^*} - Q_{t^*}) \cdot \vec{\mathbf{u}} = 0$. Which of the two facts comes from setting a derivative to zero?

31. ⋆ ⋆ ⋆ EXT. **A harder skew pair.** Using the lines of Problem 25 ($L_1 : (t, t, 2t)$, $L_2 : (1 + s, 2s, -s)$), compute $d(L_1, L_2)$ by the method of your choice.

Synthesis

32. ⋆⋆ **Two lines, one decision.** Let $L_1 : (1, 2, 0) + t(1, -1, 1)$, and let L_2 be the line through $C(0, 3, 2)$ and $D(2, 1, 4)$.

(a) Find a direction vector for L_2.

(b) Determine the relative position of L_1 and L_2. (Test *all three* coordinates.)

(c) Compute the relevant quantity: intersection point, or distance.

33. ⋆⋆ **A triangle in space.** Let $A(0, 0, 0)$, $B(4, 0, 0)$, $C(1, 3, 2)$.

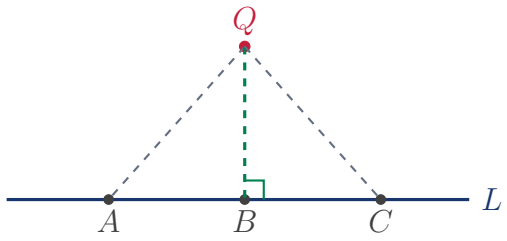
(a) Find the vector equation of line AB.

(b) Find the foot H of the altitude from C (projection of C on AB).

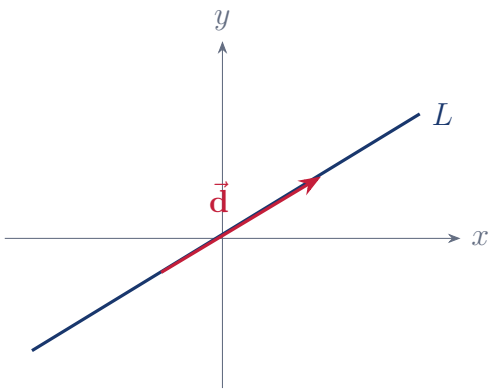
(c) Find the height $\|C - H\|$ and the area, using $\frac{1}{2} \cdot \text{base} \cdot \text{height}$.

(d) *Check:* confirm the area with $\frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\|$.

34. ⋆ **Which point is the projection?** The three points A, B, C lie on the line L. Without calculating coordinates, determine which one can be the orthogonal projection of Q onto L. Explain the single geometric property that decides the answer.



35. ⋆⋆ **Read the geometry before writing equations.** The line L below rises from left to right. The arrow shows one possible direction vector $\vec{\mathbf{d}}$.



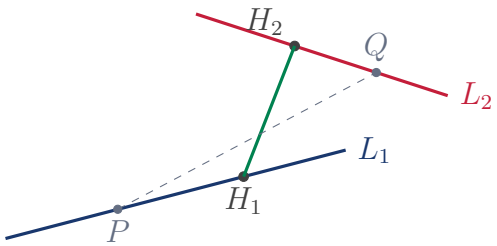
Without using coordinates from the picture, answer qualitatively.

(a) What signs must the two components of the displayed $\vec{\mathbf{d}} = (a, b)$ have?

(b) Give the possible sign patterns for a normal vector $\vec{\mathbf{n}} = (A, B)$ to L.

(c) Suppose the same line is written as $Ax + By + C = 0$ with $A > 0$. What must be the sign of B? Explain geometrically, not by memorizing a slope formula.

36. ⋆⋆ **The shortest bridge between skew lines.** The diagram represents two skew lines L_1, L_2 and points $H_1 \in L_1$, $H_2 \in L_2$. The segment H_1H_2 is perpendicular to both lines.



(a) Which segment in the picture represents $d(L_1, L_2)$?

(b) If $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are direction vectors of L_1 and L_2, write the two dot-product conditions satisfied by $\overrightarrow{H_1H_2}$.

(c) Why is the dashed segment PQ not guaranteed to represent the distance between the lines?

(d) What geometric role is played by $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ in the cross-product distance formula?

37. ⋆ **True or false** — justify each in one sentence.

(a) A line in $\mathbb{R}^3$ can always be described by one equation $Ax + By + Cz + D = 0$.

(b) Two lines with parallel direction vectors are the same line.

(c) Two non-parallel lines in $\mathbb{R}^2$ can be skew.

(d) If Q lies on L, its orthogonal projection onto L is Q.

(e) The distance between two intersecting lines is 0.

(f) In $\mathbb{R}^3$, two non-parallel lines must intersect.

Answers and key steps

1. (a) $\vec{\mathbf{d}} = (3,4)$. (b) $(x,y) = (2,-1) + t(3,4)$; $x = 2+3t$, $y = -1+4t$; $4(x-2) - 3(y+1) = 0 \Rightarrow 4x - 3y - 11 = 0$. (c) A: $8 + 3 - 11 = 0$; B: $20 - 9 - 11 = 0$.

2. (a) E.g. $(0,3)$ and $\vec{\mathbf{d}} = (-B, A) = (2,3)$. (b) $(x,y) = (0,3) + t(2,3)$; $x = 2t$, $y = 3+3t$. (c) $(2,3)\cdot(3,-2) = 0$.

3. (a) $(x,y,z) = (1,-2,3) + t(2,0,-1)$; $x = 1+2t$, $y = -2$, $z = 3-t$. (b) $(7,-2,0)$. (c) $(7,-2,0)$: yes ($t = 3$). $(3,-1,2)$: no, since $y = -2$ always.

4. (a) $3(x-2) = 0 \Rightarrow x = 2$; $-5(y-1) = 0 \Rightarrow y = 1$. (b) $a = 0$ or $b = 0$ puts a zero in a denominator. (c) It only multiplies and subtracts; it needs just $\vec{\mathbf{d}} \neq \vec{\mathbf{0}}$.

5. (a) $A(-2,-1)$, $B(1,3)$. (b) $\vec{\mathbf{d}} = (3,4)$; $(x,y) = (-2,-1) + t(3,4)$; $4x - 3y + 5 = 0$. (c) $x = 0 \Rightarrow y = \frac{5}{3} \approx 1.67$, between the ticks 1 and 2.

6. (a) L_1: $(1+t) + (1-t) + 1 = 3$. L_2: $(1+t) + (1+t) + (1-2t) = 3$. (b) Infinitely many (every direction lying in Π). (c) $x + y + z = 3$ is satisfied by all of them: it describes a plane, so it cannot single out one line. A line needs a point and a direction (or, in Cartesian form, two independent plane equations whose intersection is the line).

7. (a) $P_0 + \vec{\mathbf{d}} = (4,6)$ is on the line, but $3(4-1) = 9 \neq 16 = 4(6-2)$. (b) Correct: $4(x-1) = 3(y-2)$, i.e. $4x - 3y + 2 = 0$. The student's equation has normal $(3,-4)$, and $(3,-4)\cdot(3,4) = -7 \neq 0$: it is not perpendicular to $\vec{\mathbf{d}}$.

8. (b) $\vec{\mathbf{w}} = (2,3)$, $\vec{\mathbf{w}}\cdot\vec{\mathbf{d}} = 7$, $\vec{\mathbf{d}}\cdot\vec{\mathbf{d}} = 5$, $t = \frac{7}{5}$, $H = \left(\frac{14}{5}, \frac{12}{5}\right)$. (c) $\overrightarrow{HQ} = \left(-\frac{4}{5}, \frac{8}{5}\right)$, dot $\vec{\mathbf{d}} = 0$; $QH = \frac{4}{\sqrt{5}} \approx 1.79$.

9. $\vec{\mathbf{w}} = (6,2)$, $t = \frac{12}{10} = \frac{6}{5}$, $H = \left(\frac{6}{5}, \frac{18}{5}\right)$; $\overrightarrow{HQ} = \left(\frac{24}{5}, -\frac{8}{5}\right)$, dot $\vec{\mathbf{d}} = \frac{24}{5} - \frac{24}{5} = 0$.

10. $\vec{\mathbf{w}} = (1,-2,4)$, $\vec{\mathbf{w}}\cdot\vec{\mathbf{d}} = -5$, $\vec{\mathbf{d}}\cdot\vec{\mathbf{d}} = 3$, $t = -\frac{5}{3}$, $H = \left(-\frac{2}{3}, -\frac{2}{3}, \frac{8}{3}\right)$; $\overrightarrow{HQ} = \left(\frac{8}{3}, -\frac{1}{3}, \frac{7}{3}\right)$, dot $\vec{\mathbf{d}} = 0$.

11. $\vec{\mathbf{w}} = (-2,-1)$, $t = -\frac{4}{5}$, $H = \left(\frac{6}{5}, -\frac{3}{5}\right)$; distance $= \|H\| = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$. (Check: the line is $2x - y - 3 = 0$, and $\frac{|-3|}{\sqrt{5}}$.)

12. $Q = P_0 + k\vec{\mathbf{d}} \Rightarrow \vec{\mathbf{w}} = k\vec{\mathbf{d}} \Rightarrow t = \frac{k\vec{\mathbf{d}}\cdot\vec{\mathbf{d}}}{\vec{\mathbf{d}}\cdot\vec{\mathbf{d}}} = k \Rightarrow H = P_0 + k\vec{\mathbf{d}} = Q$.

13. $t' = \frac{\vec{\mathbf{w}}\cdot k\vec{\mathbf{d}}}{k\vec{\mathbf{d}}\cdot k\vec{\mathbf{d}}} = \frac{t}{k}$, so $H' = P_0 + \frac{t}{k}(k\vec{\mathbf{d}}) = H$. The line does not change when $\vec{\mathbf{d}}$ is rescaled, so the closest point cannot change either.

14. $\frac{|15+20-12|}{\sqrt{9+16}} = \frac{23}{5}$.

15. (a) $\vec{\mathbf{w}} = (5,3)$, $t = \frac{13}{5}$, $H = \left(\frac{31}{5}, \frac{18}{5}\right)$, $d = \left\|\left(-\frac{1}{5}, \frac{2}{5}\right)\right\| = \frac{\sqrt{5}}{5}$. (b) $x - 2y + 1 = 0$: $\frac{|6-8+1|}{\sqrt{5}} = \frac{1}{\sqrt{5}}$. (c) The shortcut is quicker once a Cartesian equation is available; in $\mathbb{R}^3$ only projection or cross product work.

16. (a) $\vec{\mathbf{w}} = (3,0,3)$, $\vec{\mathbf{w}}\cdot\vec{\mathbf{d}} = 12$, $\vec{\mathbf{d}}\cdot\vec{\mathbf{d}} = 9$, $t = \frac{4}{3}$, $H = \left(\frac{11}{3}, \frac{4}{3}, \frac{11}{3}\right)$. (b) $\overrightarrow{HQ} = \left(\frac{1}{3}, -\frac{4}{3}, \frac{1}{3}\right)$, $d = \sqrt{2}$. (c) $\frac{2}{3} - \frac{4}{3} + \frac{2}{3} = 0$.

17. (a) $\vec{\mathbf{d}} \times \vec{\mathbf{w}} = (3,0,-3)$, $\|\cdot\| = 3\sqrt{2}$, $\|\vec{\mathbf{d}}\| = 3$, so $d = \sqrt{2}$. (b) (a) $= \|\vec{\mathbf{d}}\|$ (base), (b) $= d(Q,L)$ (height), (c) $= \|\vec{\mathbf{d}} \times \vec{\mathbf{w}}\|$ (area). (c) Area = base $\times$ height gives $\|\vec{\mathbf{d}} \times \vec{\mathbf{w}}\| = \|\vec{\mathbf{d}}\|\, d(Q,L)$.

18. (a) $\vec{\mathbf{w}} = (9,0,0)$, $t = 1$, $H = (1,2,2)$, $d = \|(8,-2,-2)\| = 6\sqrt{2} \approx 8.49$. (b) The point $(1,2,2)$.

19. (a) In $\mathbb{R}^3$, $x - y = 0$ is a *plane* (containing the z-axis); the student computed the distance to that plane. (b) $\vec{\mathbf{w}} = (1,2,3)$, $t = \frac{3}{2}$, $H = \left(\frac{3}{2},\frac{3}{2},0\right)$, $d = \left\|\left(-\frac{1}{2},\frac{1}{2},3\right)\right\| = \frac{\sqrt{38}}{2} \approx 3.08$. (Cross product: $(3,-3,1)$, $\frac{\sqrt{19}}{\sqrt{2}}$.)

20. (a) Parallel distinct: same direction $(1,0,0)$; $E(0,0,1) \notin (AB)$. (b) Intersect at B: $(t,0,0) = (1,s,0) \Rightarrow t = 1, s = 0$. (c) Skew: directions $(1,0,0), (0,0,1)$ not parallel; $(t,0,0) = (1,1,s')$ fails on y. (d) Intersect at $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$: $(t,t,t) = (1-s,s,s) \Rightarrow s = t = \frac{1}{2}$, z consistent. (e) Skew: $t = 1$, $s = 1$, then z: $1 \neq 0$.

21. (a) $(-4,2) = -2(2,-1)$: parallel. $(3,0) \in L_1$? $t = 1$ gives $y = 1 \neq 0$: distinct. (b) $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \vec{\mathbf{0}}$; $(0,1,1) \in L_1$? $(1+t,t,0)$ has $z = 0 \neq 1$: distinct.

22. $t = 4 - s$ and $1 + t = -3 + 2s \Rightarrow s = \frac{8}{3}$, $t = \frac{4}{3}$; point $\left(\frac{4}{3},\frac{7}{3}\right)$. On L_2: $\left(4 - \frac{8}{3}, -3 + \frac{16}{3}\right) = \left(\frac{4}{3},\frac{7}{3}\right)$.

23. $(t,0,t) = (1,1+s,s)$: $t = 1$, $s = -1$, z: $1 \neq -1$. Skew.

24. (a) $(t,t,t) = (3-2s,-1+2s,2-s)$. Non-parallel ($\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (-3,-1,4)$). y, z: $s = 1$; then $t = 1$; x: $3 - 2 = 1$ ✓. Intersect at $(1,1,1)$. (b) Using y, z: $t = -1 + 2s$ and $t = 2 - s$, so $-1 + 2s = 2 - s \Rightarrow s = 1$ and $t = 1$. Testing x: $3 - 2(1) = 1$ ✓. Same conclusion: a common point must satisfy all three coordinate equations; once two independent equations determine the parameters, the remaining equation decides whether the lines really intersect.

25. The student skipped the third (z) equation: $2t = 4$ but $-s = -1$. The point $(2,2,4)$ is on L_1 but L_2 at $s = 1$ is $(2,2,-1)$. The lines are skew.

26. $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (0,-1,1) \neq \vec{\mathbf{0}}$. x: $t = 1$; y: $s = 1$; z: $1 = k + 1 \Rightarrow k = 0$. Intersect (at $(1,1,1)$) iff $k = 0$; skew for every other k.

27. (a) $\vec{\mathbf{w}} = (5,0)$, $t = 1$, $H = (1,2)$, $d = \|(4,-2)\| = 2\sqrt{5}$. (b) $L_1 : 2x - y = 0$; $\frac{|10|}{\sqrt{5}} = 2\sqrt{5}$.

28. (a) $d\big((AB),(EF)\big) = 1$ (parallel, E projects to A); $d\big((AB),(CG)\big) = 1$. For example, with $\vec{\mathbf{u}} = (1,0,0)$, $\vec{\mathbf{v}} = (0,0,1)$ and $C - A = (1,1,0)$, the skew-line shortcut gives $\dfrac{|(C-A)\cdot(\vec{\mathbf{u}} \times \vec{\mathbf{v}})|}{\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|} = 1$. (b) $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (-1,0,1)$, $\|\cdot\| = \sqrt{2}$, $(P_2 - P_1) = (1,0,0)$: $d = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

29. (a) $(t,t,0) = (1,s,1+s)$: $t = 1$, $s = 1$, z: $0 \neq 2$; and $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (1,-1,1) \neq \vec{\mathbf{0}}$. (b) $P_1 - P_2 = (-1,0,-1)$; $\vec{\mathbf{u}}\cdot\vec{\mathbf{u}} = \vec{\mathbf{v}}\cdot\vec{\mathbf{v}} = 2$, $\vec{\mathbf{u}}\cdot\vec{\mathbf{v}} = 1$: $-1+2t-s = 0$ and $-1+t-2s = 0$. From the first $s = 2t-1$; then $1 - 3t = 0$, so $t = \frac{1}{3}$, $s = -\frac{1}{3}$. (c) $H_1 = \left(\frac{1}{3},\frac{1}{3},0\right)$, $H_2 = \left(1,-\frac{1}{3},\frac{2}{3}\right)$, $\overrightarrow{H_1H_2} = \left(\frac{2}{3},-\frac{2}{3},\frac{2}{3}\right)$, $d = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$. (d) dots with $(1,1,0)$ and $(0,1,1)$ are both 0. (e) $\frac{|(1,0,1)\cdot(1,-1,1)|}{\sqrt{3}} = \frac{2}{\sqrt{3}}$.

30. (a) $s(t) = \frac{(P_t - P_2)\cdot\vec{\mathbf{v}}}{\vec{\mathbf{v}}\cdot\vec{\mathbf{v}}} = \frac{t-1}{2}$: affine in t. (b) $P_t - Q_t = \left(t-1,\frac{t+1}{2},-\frac{t+1}{2}\right)$, so $f^2 = (t-1)^2 + \frac{(t+1)^2}{2} = \frac{3}{2}t^2 - t + \frac{3}{2}$. (c) $f^{2\prime} = 3t - 1 = 0 \Rightarrow t^* = \frac{1}{3}$; $f^2 = \frac{4}{3}$, $f = \frac{2}{\sqrt{3}}$ — same as Problem 29. (d) $(P_t - Q_t)\cdot(0,1,1) = \frac{t+1}{2} - \frac{t+1}{2} = 0$ always (definition of projection); at t^*: $\left(-\frac{2}{3},\frac{2}{3},-\frac{2}{3}\right)\cdot(1,1,0) = 0$, and this one comes from setting the derivative to zero.

31. $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (-5,3,1)$, $\|\cdot\| = \sqrt{35}$, $P_2 - P_1 = (1,0,0)$: $d = \frac{5}{\sqrt{35}} = \frac{\sqrt{35}}{7} \approx 0.85$.

32. (a) $D - C = (2,-2,2) = 2(1,-1,1)$, so take $\vec{\mathbf{v}} = (1,-1,1)$. (b) Parallel to L_1. $C \in L_1$? $1 + t = 0 \Rightarrow t = -1$; $y = 2 - (-1) = 3$ ✓; $z = -1 \neq 2$. Parallel and distinct. (c) $\vec{\mathbf{w}} = C - P_1 = (-1,1,2)$, $\vec{\mathbf{w}}\cdot\vec{\mathbf{u}} = 0 \Rightarrow t = 0$, $H = (1,2,0)$, $d = \|(-1,1,2)\| = \sqrt{6}$.

33. (a) $(x, y, z) = t(4, 0, 0)$. (b) $t = \frac{4}{16}$, $H = (1, 0, 0)$. (c) height $= \|(0, 3, 2)\| = \sqrt{13}$, area $= \frac{1}{2} \cdot 4 \cdot \sqrt{13} = 2\sqrt{13}$. (d) $\overrightarrow{AB} \times \overrightarrow{AC} = (0, -8, 12)$, norm $4\sqrt{13}$, half is $2\sqrt{13}$.

34. B. The defining property of the orthogonal projection is $QH \perp L$; only QB is perpendicular to the line.

35. (a) For the displayed direction, $a > 0$ and $b > 0$. (b) A normal must have components of opposite signs: (A, B) can have sign pattern $(+, -)$ or $(-, +)$. (c) If $A > 0$, then $B < 0$. Geometrically, (A, B) must be perpendicular to a direction vector whose two components have the same sign, so the two terms in $Aa + Bb = 0$ must cancel.

36. (a) $\overline{H_1H_2}$. (b) $\overrightarrow{H_1H_2} \cdot \vec{\mathbf{u}} = 0$ and $\overrightarrow{H_1H_2} \cdot \vec{\mathbf{v}} = 0$. (c) P and Q are arbitrary points; a segment joining arbitrary points on the two lines need not be perpendicular to both and therefore need not have minimum length. (d) $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ gives a direction perpendicular to both lines. The shortest connecting segment is parallel to this common-normal direction, and the scalar triple product extracts the component of $P_2 - P_1$ along it.

37. (a) False: that is a plane. (b) False: they may be distinct parallel lines. (c) False: in $\mathbb{R}^2$ non-parallel lines always meet. (d) True. (e) True. (f) False: they may be skew.