Chapter 3 Summary

Equations, Projection, Distance, and Relative Position

Mohamed Nouidha

The one idea behind the whole chapter

Everything in this chapter comes from **perpendicularity**: detected with the **dot product** ($\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}=0$) and, in $\mathbb{R}^3$, with the **cross product** ($\vec{\mathbf{a}}\times\vec{\mathbf{b}}$ is perpendicular to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$). No matrices or linear-system methods are needed: every system below is solved by substitution.

1 Describing a Line

A line is determined by **two distinct points** A, B, or by **one point** P_0 and a **direction vector** $\vec{\mathbf{d}}\neq\vec{\mathbf{0}}$. The two descriptions are linked by $\vec{\mathbf{d}}=\overrightarrow{AB}=B-A$.

The three forms of a line in $\mathbb{R}^2$

With $P_0=(x_0,y_0)$ and $\vec{\mathbf{d}}=(a,b)$:

Vector form:	$(x,y)=(x_0,y_0)+t(a,b), \quad t\in\mathbb{R}$
Parametric form:	$x=x_0+ta, \quad y=y_0+tb$
Cartesian form:	$Ax+By+C=0$, with normal vector $\vec{\mathbf{n}}=(A,B)$

Conversions.

- Direction → normal: $\vec{\mathbf{d}}=(a,b) \Rightarrow \vec{\mathbf{n}}=(b,-a)$, then write $\vec{\mathbf{n}}\cdot\overrightarrow{P_0P}=0$, i.e. $b(x-x_0)-a(y-y_0)=0$.
- Normal → direction: $Ax+By+C=0 \Rightarrow \vec{\mathbf{d}}=(-B,A)$.

Be careful

- The direction vector is **not unique**: any $k\vec{\mathbf{d}}$ with $k\neq0$ describes the same line. The same holds for the base point P_0 (any point of the line works).
- In $\mathbb{R}^3$ a single equation $Ax+By+Cz+D=0$ is a **plane**, not a line. A line in space is described in **vector/parametric form** only:

$$(x,y,z)=(x_0,y_0,z_0)+t(a,b,c) \iff \begin{cases} x=x_0+ta \\ y=y_0+tb \\ z=z_0+tc \end{cases} \qquad t\in\mathbb{R}.$$

Example 1 — From two points to all three forms

Line through $A(2,-1)$ and $B(5,3)$.

1. Direction: $\vec{\mathbf{d}} = B - A = (3,4)$.
2. Vector form: $(x,y) = (2,-1) + t(3,4)$. Parametric: $x = 2+3t,\ y = -1+4t$.
3. Normal: $\vec{\mathbf{n}} = (4,-3)$, so $4(x-2) - 3(y+1) = 0$, i.e. $\boxed{4x - 3y - 11 = 0}$.
4. Check with B: $4(5) - 3(3) - 11 = 0$ ✓

Example 2 — From Cartesian to parametric

$L:\ 2x + 5y - 10 = 0$. Normal $\vec{\mathbf{n}} = (2,5)$, so a direction is $\vec{\mathbf{d}} = (-5,2)$. A point: set $x = 0$, then $5y = 10$, so $P_0 = (0,2)$.

$$L:\ (x,y) = (0,2) + t(-5,2).$$

Check: $2(-5t) + 5(2+2t) - 10 = -10t + 10 + 10t - 10 = 0$ for every t ✓

Example 3 — A line in $\mathbb{R}^3$ and point membership

Line through $A(1,2,-1)$ and $B(3,-1,4)$: $\vec{\mathbf{d}} = (2,-3,5)$, so $(x,y,z) = (1+2t,\ 2-3t,\ -1+5t)$.
Does $C(5,-4,9)$ lie on it? The x-coordinate gives $1 + 2t = 5 \Rightarrow t = 2$. Substitute: $y = 2 - 3(2) = -4$ ✓, $z = -1 + 5(2) = 9$ ✓. **Yes**, C is the point with $t = 2$.
(If even one coordinate had failed with $t = 2$, the point would *not* be on the line.)

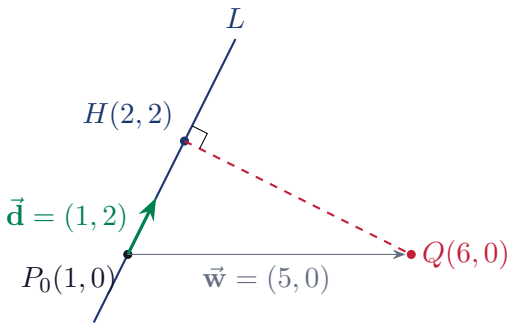
2 Orthogonal Projection of a Point onto a Line

Projection formula (same in $\mathbb{R}^2$ and $\mathbb{R}^3$)

Let L pass through P_0 with direction $\vec{\mathbf{d}}$, and let Q be a point. Set $\vec{\mathbf{w}} = \overrightarrow{P_0Q} = Q - P_0$. The **orthogonal projection** of Q onto L (the point of L closest to Q) is

$$\boxed{H = P_0 + t\,\vec{\mathbf{d}}, \qquad t = \frac{\vec{\mathbf{w}} \cdot \vec{\mathbf{d}}}{\vec{\mathbf{d}} \cdot \vec{\mathbf{d}}}}$$

Why: H is on L, so $H = P_0 + t\vec{\mathbf{d}}$; and $\overrightarrow{HQ} \perp \vec{\mathbf{d}}$ gives $(\vec{\mathbf{w}} - t\vec{\mathbf{d}}) \cdot \vec{\mathbf{d}} = 0$, which is solved for t.



Example 4 — Projection in $\mathbb{R}^2$ (the figure above)

Project $Q(6,0)$ onto $L:\ (x,y) = (1,0) + t(1,2)$.

1. $\vec{\mathbf{w}} = Q - P_0 = (5,0)$; $\vec{\mathbf{w}} \cdot \vec{\mathbf{d}} = 5$; $\vec{\mathbf{d}} \cdot \vec{\mathbf{d}} = 1 + 4 = 5$; $t = 1$.
2. $H = (1,0) + 1(1,2) = (2,2)$.
3. Verify: $\overrightarrow{HQ} = (4,-2)$ and $(4,-2) \cdot (1,2) = 4 - 4 = 0$ ✓

Example 5 — Projection in $\mathbb{R}^3$

Project $Q(3,1,5)$ onto $L:\ (x,y,z) = (1,0,1) + t(1,2,2)$.

1. $\vec{\mathbf{w}} = (2,1,4)$; $\vec{\mathbf{w}} \cdot \vec{\mathbf{d}} = 2 + 2 + 8 = 12$; $\vec{\mathbf{d}} \cdot \vec{\mathbf{d}} = 1 + 4 + 4 = 9$; $t = \frac{12}{9} = \frac{4}{3}$.
2. $H = (1,0,1) + \frac{4}{3}(1,2,2) = \left(\frac{7}{3},\ \frac{8}{3},\ \frac{11}{3}\right)$.
3. Verify: $\overrightarrow{HQ} = \left(\frac{2}{3}, -\frac{5}{3}, \frac{4}{3}\right)$ and $\overrightarrow{HQ} \cdot \vec{\mathbf{d}} = \frac{2-10+8}{3} = 0$ ✓

3 Distance from a Point to a Line

Three ways to compute $d(Q,L)$

Projection ($\mathbb{R}^2$ and $\mathbb{R}^3$): $d(Q,L) = \|Q - H\|$, H = projection of Q onto L

Cartesian shortcut ($\mathbb{R}^2$ only): $d(Q,L) = \dfrac{|Ax_Q + By_Q + C|}{\sqrt{A^2 + B^2}}$ for $L: Ax + By + C = 0$

Cross-product shortcut ($\mathbb{R}^3$): $d(Q,L) = \dfrac{\|\vec{\mathbf{d}} \times \vec{\mathbf{w}}\|}{\|\vec{\mathbf{d}}\|}$, $\vec{\mathbf{w}} = Q - P_0$

The cross-product formula is "area of the parallelogram spanned by $\vec{\mathbf{d}}$ and $\vec{\mathbf{w}}$, divided by its base $\|\vec{\mathbf{d}}\|$" (= its height).

Be careful

The Cartesian formula needs the line in the form $Ax + By + C = 0$ (everything on one side) and is valid **only in** $\mathbb{R}^2$. Do not forget the absolute value.

Example 6 — Same distance, two methods ($\mathbb{R}^2$)

Distance from $Q(6,0)$ to $L:\ (1,0) + t(1,2)$ (Example 4).

- **Projection:** $H = (2,2)$, so $d = \|(4,-2)\| = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$.
- **Cartesian:** $\vec{\mathbf{n}} = (2,-1)$ gives $2(x-1) - (y-0) = 0$, i.e. $2x - y - 2 = 0$. Then

$$d = \frac{|2(6) - 0 - 2|}{\sqrt{4+1}} = \frac{10}{\sqrt{5}} = 2\sqrt{5}\ \checkmark$$

Example 7 — Same distance, two methods ($\mathbb{R}^3$)

Distance from $Q(3,1,5)$ to $L:\ (1,0,1) + t(1,2,2)$ (Example 5).

- **Projection:** $\|Q - H\| = \left\|\left(\frac{2}{3}, -\frac{5}{3}, \frac{4}{3}\right)\right\| = \frac{\sqrt{4+25+16}}{3} = \frac{\sqrt{45}}{3} = \sqrt{5}$.
- **Cross product:** $\vec{\mathbf{d}} \times \vec{\mathbf{w}} = (1,2,2) \times (2,1,4) = (2\cdot 4 - 2\cdot 1,\ 2\cdot 2 - 1\cdot 4,\ 1\cdot 1 - 2\cdot 2) = (6,0,-3)$.

$$d = \frac{\|(6,0,-3)\|}{\|(1,2,2)\|} = \frac{\sqrt{45}}{3} = \sqrt{5}\ \checkmark$$

4 Relative Position of Two Lines

Let L_1 pass through P_1 with direction $\vec{\mathbf{u}}$, and L_2 through P_2 with direction $\vec{\mathbf{v}}$.

Relationship	Directions	Common points
Equal (coincident)	parallel	all
Parallel & distinct	parallel	none
Intersecting	not parallel	exactly one
Skew ($\mathbb{R}^3$ only)	not parallel	none

Decision procedure

Step 1 — Are the directions parallel?

- In $\mathbb{R}^2$: $\vec{\mathbf{u}} \parallel \vec{\mathbf{v}} \iff u_1 v_2 - u_2 v_1 = 0$.
- In $\mathbb{R}^3$: $\vec{\mathbf{u}} \parallel \vec{\mathbf{v}} \iff \vec{\mathbf{u}} \times \vec{\mathbf{v}} = \vec{\mathbf{0}}$.
- In both: $\vec{\mathbf{u}} \parallel \vec{\mathbf{v}} \iff \vec{\mathbf{v}} = k\vec{\mathbf{u}}$ for some scalar k.

Step 2a — If parallel: test whether P_2 lies on L_1 (equivalently, whether $\overrightarrow{P_1P_2} \parallel \vec{\mathbf{u}}$). Yes $\Rightarrow$ **equal**; no $\Rightarrow$ **parallel and distinct**.

Step 2b — If not parallel: write $P_1 + t\vec{\mathbf{u}} = P_2 + s\vec{\mathbf{v}}$ component by component.

- In $\mathbb{R}^2$: two equations, two unknowns; solve by substitution. The lines **always intersect**.
- In $\mathbb{R}^3$: three equations, two unknowns. Solve **two** of them for t, s, then test the **third**. Holds $\Rightarrow$ **intersecting**; fails $\Rightarrow$ **skew**.

Example 8 — Intersecting lines in $\mathbb{R}^2$

$L_1 : \ (1,1) + t(2,-1), \quad L_2 : \ (0,-3) + s(1,1)$.
Step 1: $u_1v_2 - u_2v_1 = 2(1) - (-1)(1) = 3 \neq 0$: not parallel, so they intersect.
Step 2: $x : \ 1 + 2t = s$ and $y : \ 1 - t = -3 + s$. Substituting $s = 1 + 2t$ into the second: $1 - t = -2 + 2t \Rightarrow t = 1$, hence $s = 3$.
Intersection: $(1,1) + 1(2,-1) = (3,0)$. Check on L_2: $(0,-3) + 3(1,1) = (3,0)$ ✓

Example 9 — Parallel and distinct lines in $\mathbb{R}^3$

$L_1 : \ (1,0,2) + t(1,-1,2), \quad L_2 : \ (0,1,1) + s(-2,2,-4)$.
Step 1: $\vec{\mathbf{v}} = -2\vec{\mathbf{u}}$, so $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \vec{\mathbf{0}}$: parallel.
Step 2a: is $P_2(0,1,1)$ on L_1? From x: $1 + t = 0 \Rightarrow t = -1$. Then $y = 0 - (-1) = 1$ ✓ but $z = 2 + 2(-1) = 0 \neq 1$ ✗. **Parallel and distinct.**

Example 10 — Intersecting lines in $\mathbb{R}^3$

$L_1 : \ (1,1,0) + t(1,0,1) = (1+t,\ 1,\ t), \quad L_2 : \ (2,3,1) + s(0,1,0) = (2,\ 3+s,\ 1)$.
Step 1: $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (1,0,1) \times (0,1,0) = (-1,0,1) \neq \vec{\mathbf{0}}$: not parallel.
Step 2b: $x : \ 1 + t = 2 \Rightarrow t = 1; \quad y : \ 1 = 3 + s \Rightarrow s = -2; \quad$ test $z : \ t = 1$, and indeed $t = 1$ ✓
The lines **intersect** at $(1+1,\ 1,\ 1) = (2,1,1)$. Check on L_2: $(2,\ 3-2,\ 1) = (2,1,1)$ ✓

Example 11 — Skew lines

$L_1 : \ (0,0,0) + t(1,0,0) = (t,0,0)$ (the x-axis), $\quad L_2 : \ (0,1,3) + s(0,1,1) = (0,\ 1+s,\ 3+s)$.
Step 1: $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (1,0,0) \times (0,1,1) = (0,-1,1) \neq \vec{\mathbf{0}}$: not parallel.
Step 2b: $x : \ t = 0; \quad y : \ 0 = 1 + s \Rightarrow s = -1; \quad$ test $z : \ 0 = 3 + s$, but $3 + (-1) = 2 \neq 0$ **fails**.
No common point and not parallel: the lines are **skew**.

5 Distance Between Two Lines

Distance by case

Equal or intersecting:	$d(L_1, L_2) = 0$
Parallel & distinct:	$d(L_1, L_2) = d(P_2, L_1)$ (any point of L_2 works)
Skew:	$d(L_1, L_2) = \dfrac{\lvert(P_2 - P_1) \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}})\rvert}{\lVert\vec{\mathbf{u}} \times \vec{\mathbf{v}}\rVert}$

Why the skew formula: $\vec{\mathbf{n}} = \vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is perpendicular to *both* lines, so it points along their common perpendicular. The distance is the length of the "shadow" of $\overrightarrow{P_1P_2}$ on $\vec{\mathbf{n}}$, i.e. $|\overrightarrow{P_1P_2} \cdot \vec{\mathbf{n}}|/\|\vec{\mathbf{n}}\|$.

Example 12 — Distance between parallel lines (Example 9)

Use $P_2(0, 1, 1)$ and $L_1 : \ (1, 0, 2) + t(1, -1, 2)$. Then $\vec{\mathbf{w}} = P_2 - P_1 = (-1, 1, -1)$.

- **Projection:** $t = \dfrac{\vec{\mathbf{w}} \cdot \vec{\mathbf{u}}}{\vec{\mathbf{u}} \cdot \vec{\mathbf{u}}} = \dfrac{-1 - 1 - 2}{6} = -\dfrac{2}{3}$, $H = (1, 0, 2) - \frac{2}{3}(1, -1, 2) = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$,

$$d = \|P_2 - H\| = \left\|\left(-\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right)\right\| = \frac{\sqrt{3}}{3}.$$

- **Cross product:** $\vec{\mathbf{u}} \times \vec{\mathbf{w}} = (1, -1, 2) \times (-1, 1, -1) = (-1, -1, 0)$, so $d = \dfrac{\sqrt{2}}{\sqrt{6}} = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$ ✓

Example 13 — Distance between skew lines (Example 11)

$P_2 - P_1 = (0, 1, 3)$ and $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (0, -1, 1)$:

$$d(L_1, L_2) = \frac{|(0, 1, 3) \cdot (0, -1, 1)|}{\|(0, -1, 1)\|} = \frac{|0 - 1 + 3|}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

Sanity check. A point of L_2 is $(0, 1 + s, 3 + s)$; its distance to the x-axis is $\sqrt{(1 + s)^2 + (3 + s)^2}$, smallest at $s = -2$, where it equals $\sqrt{1 + 1} = \sqrt{2}$ ✓

A useful by-product

For two **non-parallel** lines in $\mathbb{R}^3$, the skew-distance numerator $(P_2 - P_1) \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}})$ is a scalar triple product. It equals 0 exactly when the lines are coplanar, i.e. when they **intersect**. In Example 10: $(1, 2, 1) \cdot (-1, 0, 1) = 0$ (intersecting); in Example 11: $2 \neq 0$ (skew). This gives a second way to check Step 2b.

6 Quick Reference

Question	**Tool**
Describe a line	$P_0 + t\vec{\mathbf{d}}$; in $\mathbb{R}^2$ also $Ax + By + C = 0$ with $\vec{\mathbf{n}} = (A, B) \perp \vec{\mathbf{d}}$
Is a point on a line?	Find t from one coordinate, check the others
Closest point H on L to Q	$H = P_0 + t\vec{\mathbf{d}},\ t = \dfrac{\vec{\mathbf{w}} \cdot \vec{\mathbf{d}}}{\vec{\mathbf{d}} \cdot \vec{\mathbf{d}}},\ \vec{\mathbf{w}} = Q - P_0$
Distance point–line	$\|Q - H\|$; $\mathbb{R}^2$: $\dfrac{\lvert Ax_Q + By_Q + C\rvert}{\sqrt{A^2 + B^2}}$; $\mathbb{R}^3$: $\dfrac{\|\vec{\mathbf{d}} \times \vec{\mathbf{w}}\|}{\|\vec{\mathbf{d}}\|}$
Parallel directions?	$\mathbb{R}^2$: $u_1 v_2 - u_2 v_1 = 0$; $\mathbb{R}^3$: $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \vec{\mathbf{0}}$
Equal or distinct (if parallel)?	Is P_2 on L_1?
Intersecting or skew (if not parallel)?	Solve two component equations, test the third
Distance between two lines	0; $d(P_2, L_1)$; or $\dfrac{\lvert(P_2 - P_1) \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}})\rvert}{\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|}$