Chapter 3 Solutions

Lines in the Plane and in Space

Equations, Orthogonal Projection, Distances, Relative Position, and Skew Lines

Collège Sainte-Anne | Fall 2026

How to use these solutions. Try each problem first. Every solution follows the same three habits: *sketch or predict*, *compute*, then *check* (a substitution, a dot product, or a second method). No matrices or systems techniques are used: whenever two equations in two parameters appear, we solve by substitution.

Defining a Line

Problem 1 ⋆ Two points

(a) A direction vector is any non-zero vector joining two points of the line:

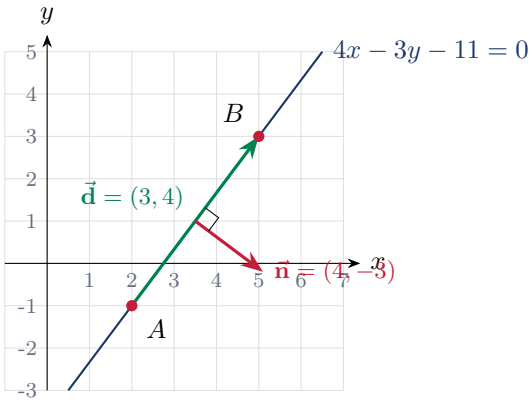
$$\vec{\mathbf{d}} = \overrightarrow{AB} = B - A = (5-2,\ 3-(-1)) = (3,4).$$

(b) Take $P_0 = A(2,-1)$ and $\vec{\mathbf{d}} = (3,4)$.

- Vector form: $(x,y) = (2,-1) + t(3,4),\ \ t \in \mathbb{R}$.
- Parametric form: $x = 2+3t, \quad y = -1+4t$.
- Cartesian form: with $a=3$, $b=4$, the formula $b(x-x_0) - a(y-y_0) = 0$ gives

$$4(x-2) - 3(y+1) = 0 \iff \boxed{4x - 3y - 11 = 0}.$$

The normal vector is $\vec{\mathbf{n}} = (4,-3)$, and indeed $\vec{\mathbf{n}} \cdot \vec{\mathbf{d}} = 12 - 12 = 0$.



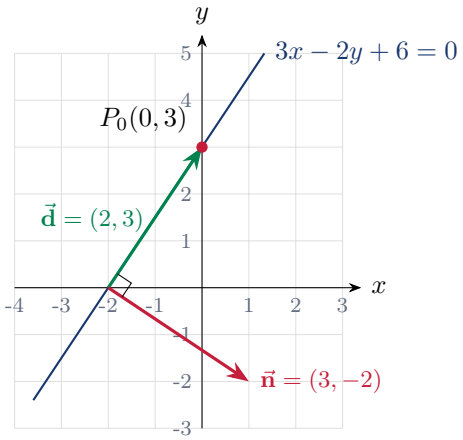
(c) $A(2,-1)$: $4(2) - 3(-1) - 11 = 8 + 3 - 11 = 0$ ✓
$B(5,3)$: $4(5) - 3(3) - 11 = 20 - 9 - 11 = 0$ ✓

Problem 2 ⋆ From Cartesian to vector

(a) *A point.* Set $x = 0$: $-2y + 6 = 0 \Rightarrow y = 3$, so $P_0 = (0, 3)$ is on the line.
A direction. The normal vector is $\vec{\mathbf{n}} = (A, B) = (3, -2)$; a vector perpendicular to it is $\vec{\mathbf{d}} = (-B, A) = (2, 3)$.

(b) $(x, y) = (0, 3) + t(2, 3)$, i.e. $x = 2t, \quad y = 3 + 3t$.

(c) $\vec{\mathbf{d}} \cdot \vec{\mathbf{n}} = (2)(3) + (3)(-2) = 6 - 6 = 0$ ✓.
Extra check: at $t = -1$ we get $(-2, 0)$, and $3(-2) - 2(0) + 6 = 0$ ✓.



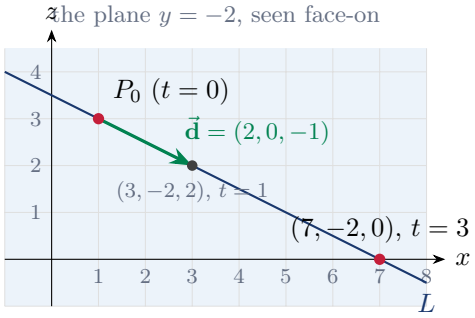
Problem 3 ⋆ A line in space

(a) $(x, y, z) = (1, -2, 3) + t(2, 0, -1)$, i.e. $x = 1 + 2t, \quad y = -2, \quad z = 3 - t$.
Note that $y = -2$ for every t: the line lies in the plane $y = -2$, because the y-component of $\vec{\mathbf{d}}$ is 0.

(b) $t = 3$: $(1 + 6,\ -2,\ 3 - 3) = (7, -2, 0)$.

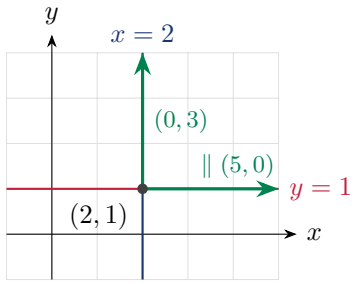
(c) *Method:* use one coordinate to find the only possible t, then test the others.

- $(7, -2, 0)$: x gives $1 + 2t = 7 \Rightarrow t = 3$; then $y = -2$ ✓, $z = 3 - 3 = 0$ ✓. **On the line.**
- $(3, -1, 2)$: x gives $t = 1$; then $y = -2 \neq -1$. **Not on the line.** (Even before solving: every point of the line has $y = -2$.)



$(3, -1, 2)$ has $y = -1$: it is not even in this plane. Its "shadow" $(3, -2, 2)$ is on L, at $t = 1$.

Problem 4 ⋆ Vertical and horizontal lines



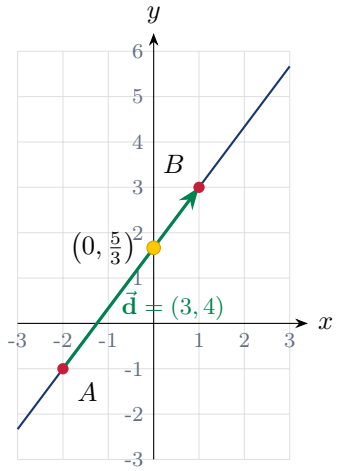
(a) With $(x_0, y_0) = (2, 1)$:

- $\vec{\mathbf{d}} = (0, 3)$: $3(x-2) - 0\,(y-1) = 0 \iff \boxed{x = 2}$ (vertical line).
- $\vec{\mathbf{d}} = (5, 0)$: $0\,(x-2) - 5(y-1) = 0 \iff \boxed{y = 1}$ (horizontal line).

(b) The fraction form would read $\dfrac{x-2}{0} = \dfrac{y-1}{3}$ and $\dfrac{x-2}{5} = \dfrac{y-1}{0}$: a division by zero appears in each case. The form is meaningless as soon as one component of $\vec{\mathbf{d}}$ is 0.

(c) The form $b(x - x_0) - a(y - y_0) = 0$ only *multiplies* by a and b; it never divides. It is valid as soon as $\vec{\mathbf{d}} \neq \vec{\mathbf{0}}$ (at least one of a, b is non-zero), which is exactly the condition for $\vec{\mathbf{d}}$ to be a direction.

Problem 5 ⋆⋆ Read from the graph



(a) $A(-2, -1)$ and $B(1, 3)$.

(b) $\vec{\mathbf{d}} = B - A = (3, 4)$.
Vector: $(x, y) = (-2, -1) + t(3, 4)$. Parametric: $x = -2 + 3t,\ y = -1 + 4t$.
Cartesian:

$$4(x+2) - 3(y+1) = 0 \iff \boxed{4x - 3y + 5 = 0}.$$

Check: B: $4 - 9 + 5 = 0$ ✓.

(c) On the y-axis $x = 0$: $-3y + 5 = 0 \Rightarrow y = \frac{5}{3} \approx 1.67$. The predicted intercept is $(0, \frac{5}{3})$, between the ticks 1 and 2, closer to 2, which is what the graph shows.

Problem 6 ⋆⋆ EXT. Why one equation is not enough in $\mathbb{R}^3$

(a) L_1: $(x, y, z) = (1, 1, 1) + t(1, -1, 0) = (1 + t,\ 1 - t,\ 1)$. Substituting:

$$(1 + t) + (1 - t) + 1 = 3 \quad \text{for every } t.\checkmark$$

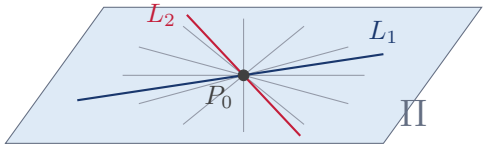
L_2: $(x, y, z) = (1 + t,\ 1 + t,\ 1 - 2t)$. Substituting:

$$(1 + t) + (1 + t) + (1 - 2t) = 3 + (t + t - 2t) = 3 \quad \text{for every } t.\checkmark$$

The reason: both direction vectors satisfy $a+b+c=0$, i.e. they are perpendicular to the normal $\vec{\mathbf{n}}=(1,1,1)$ of Π.

(b) Any vector $\vec{\mathbf{d}}=(a,b,c)\neq\vec{\mathbf{0}}$ with $a+b+c=0$ (for example $(1,0,-1)$, $(2,-1,-1)$, $(0,1,-1)$, …) gives a line through P_0 contained in Π. There are **infinitely many** such lines, a whole "fan" of them.

(c) All of these lines satisfy the single equation $x+y+z=3$, so that equation cannot tell them apart: it describes the whole plane Π, not a line. To pin down one line in $\mathbb{R}^3$ we need a point and a direction (vector/parametric form), or *two* independent plane equations whose intersection is the line.



every grey line also lies in Π

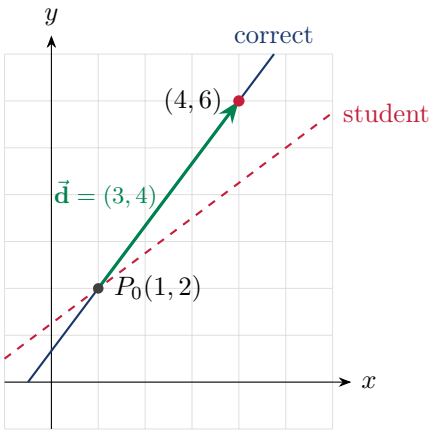
Problem 7 ⋆⋆ EXT. Find the error

(a) $P_0+\vec{\mathbf{d}}=(1,2)+(3,4)=(4,6)$ is certainly on the line. In the student's equation: left side $3(4-1)=9$, right side $4(6-2)=16$, and $9\neq16$. So the equation is wrong.

(b) With $a=3$, $b=4$: $b(x-x_0)-a(y-y_0)=0$ gives

$$4(x-1)-3(y-2)=0 \iff \boxed{4x-3y+2=0}.$$

Check: $(4,6)$: $16-18+2=0$ ✓; $(1,2)$: $4-6+2=0$ ✓.
The mistake. The student's equation is $3x-4y+5=0$, whose normal is $(3,-4)$. But $(3,-4)\cdot(3,4)=9-16=-7\neq0$: this normal is *not* perpendicular to $\vec{\mathbf{d}}$. The student put the components of $\vec{\mathbf{d}}$ in the wrong places; the correct normal is $(b,-a)=(4,-3)$. Geometrically, the student's line still passes through P_0 but has direction $(4,3)$ instead of $(3,4)$, i.e. it is the mirror image of L across the line of slope 1 through P_0.

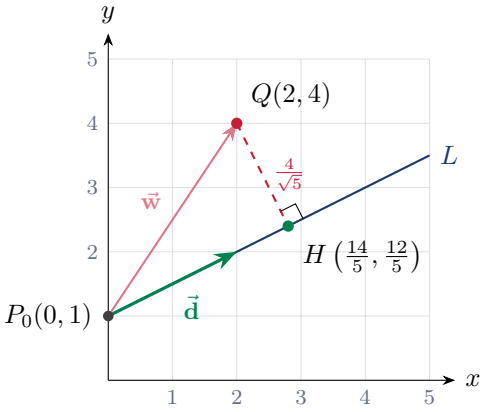


Orthogonal Projection of a Point on a Line

Recipe used throughout. $\vec{\mathbf{w}} = Q - P_0$, $t = \dfrac{\vec{\mathbf{w}} \cdot \vec{\mathbf{d}}}{\vec{\mathbf{d}} \cdot \vec{\mathbf{d}}}$, $H = P_0 + t\vec{\mathbf{d}}$, then check $\overrightarrow{HQ} \cdot \vec{\mathbf{d}} = 0$.

Problem 8 ⋆ Estimate, then compute

(a) Dropping a perpendicular from $Q(2,4)$ to L by eye, the foot lands at roughly $H \approx (2.8,\ 2.4)$, and QH looks like slightly less than 2 units, say $QH \approx 1.8$.



(b) $\vec{\mathbf{w}} = Q - P_0 = (2,4) - (0,1) = (2,3)$, $\vec{\mathbf{w}} \cdot \vec{\mathbf{d}} = 4 + 3 = 7$, $\vec{\mathbf{d}} \cdot \vec{\mathbf{d}} = 4 + 1 = 5$, so $t = \frac{7}{5}$ and

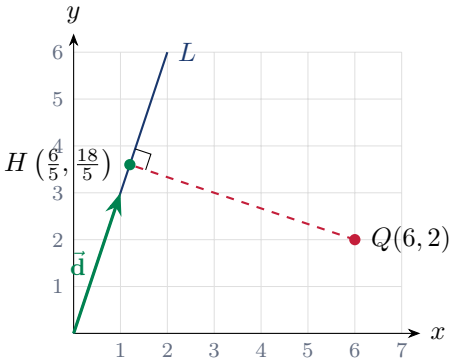
$$H = (0,1) + \tfrac{7}{5}(2,1) = \left(\tfrac{14}{5},\ \tfrac{12}{5}\right) = (2.8,\ 2.4).$$

(c) $\overrightarrow{HQ} = Q - H = \left(2 - \frac{14}{5},\ 4 - \frac{12}{5}\right) = \left(-\frac{4}{5},\ \frac{8}{5}\right)$ and $\overrightarrow{HQ} \cdot \vec{\mathbf{d}} = -\frac{8}{5} + \frac{8}{5} = 0$ ✓.
$QH = \left\|\left(-\frac{4}{5}, \frac{8}{5}\right)\right\| = \frac{1}{5}\sqrt{16+64} = \frac{\sqrt{80}}{5} = \frac{4}{\sqrt{5}} \approx 1.79$, consistent with the estimate.

Problem 9 ⋆ Projection in the plane

$P_0 = (0,0)$, $\vec{\mathbf{d}} = (1,3)$, $\vec{\mathbf{w}} = Q - P_0 = (6,2)$.
$\vec{\mathbf{w}} \cdot \vec{\mathbf{d}} = 6 + 6 = 12$, $\vec{\mathbf{d}} \cdot \vec{\mathbf{d}} = 1 + 9 = 10$, $t = \frac{12}{10} = \frac{6}{5}$, so

$$H = \tfrac{6}{5}(1,3) = \left(\tfrac{6}{5},\ \tfrac{18}{5}\right).$$



Check. $\overrightarrow{HQ} = \left(6 - \frac{6}{5},\ 2 - \frac{18}{5}\right) = \left(\frac{24}{5},\ -\frac{8}{5}\right)$, $\overrightarrow{HQ} \cdot \vec{\mathbf{d}} = \frac{24}{5} - \frac{24}{5} = 0$ ✓.
(Bonus: $QH = \frac{8}{5}\sqrt{9+1} = \frac{8\sqrt{10}}{5} \approx 5.06$.)

Problem 10 ⋆⋆ Projection in space

$\vec{\mathbf{w}} = Q - P_0 = (2,-1,5) - (1,1,1) = (1,-2,4)$.
$\vec{\mathbf{w}} \cdot \vec{\mathbf{d}} = 1 - 2 - 4 = -5$, $\vec{\mathbf{d}} \cdot \vec{\mathbf{d}} = 1 + 1 + 1 = 3$, $t = -\frac{5}{3}$.

$$H = (1,1,1) - \tfrac{5}{3}(1,1,-1) = \left(-\tfrac{2}{3},\ -\tfrac{2}{3},\ \tfrac{8}{3}\right).$$

Check. $\overrightarrow{HQ} = \left(2 + \frac{2}{3},\ -1 + \frac{2}{3},\ 5 - \frac{8}{3}\right) = \left(\frac{8}{3},\ -\frac{1}{3},\ \frac{7}{3}\right)$ and $\overrightarrow{HQ} \cdot \vec{\mathbf{d}} = \frac{8}{3} - \frac{1}{3} - \frac{7}{3} = 0$ ✓.

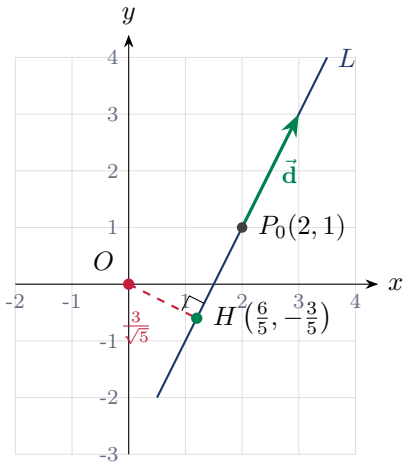
Remark. $t < 0$ simply means that H lies on the side of P_0 opposite to $\vec{\mathbf{d}}$: the angle between $\vec{\mathbf{w}}$ and $\vec{\mathbf{d}}$ is obtuse ($\vec{\mathbf{w}} \cdot \vec{\mathbf{d}} < 0$).

Problem 11 ⋆⋆ Closest point

The closest point of L to Q is the orthogonal projection H of Q on L.
$P_0 = (2,1)$, $\vec{\mathbf{d}} = (1,2)$, $Q = (0,0)$, so $\vec{\mathbf{w}} = Q - P_0 = (-2,-1)$.
$\vec{\mathbf{w}} \cdot \vec{\mathbf{d}} = -2-2 = -4$, $\vec{\mathbf{d}} \cdot \vec{\mathbf{d}} = 5$, $t = -\frac{4}{5}$:

$$H = (2,1) - \tfrac{4}{5}(1,2) = \left(\tfrac{6}{5},\ -\tfrac{3}{5}\right),$$

$$d(O,L) = \|H\| = \tfrac{1}{5}\sqrt{36+9} = \tfrac{3}{\sqrt{5}} = \tfrac{3\sqrt{5}}{5} \approx 1.34.$$



Check. $\overrightarrow{HO} \cdot \vec{\mathbf{d}} = \left(-\frac{6}{5}, \frac{3}{5}\right) \cdot (1,2) = -\frac{6}{5} + \frac{6}{5} = 0$ ✓.
Second method: L has normal $(2,-1)$: $2(x-2) - (y-1) = 0 \iff 2x - y - 3 = 0$, and $d(O,L) = \frac{|0-0-3|}{\sqrt{4+1}} = \frac{3}{\sqrt{5}}$ ✓.

Problem 12 ⋆⋆ EXT. A point already on the line

If $Q \in L$, there is a real number k with $Q = P_0 + k\vec{\mathbf{d}}$. Then $\vec{\mathbf{w}} = Q - P_0 = k\vec{\mathbf{d}}$ and

$$t = \frac{\vec{\mathbf{w}} \cdot \vec{\mathbf{d}}}{\vec{\mathbf{d}} \cdot \vec{\mathbf{d}}} = \frac{k\,(\vec{\mathbf{d}} \cdot \vec{\mathbf{d}})}{\vec{\mathbf{d}} \cdot \vec{\mathbf{d}}} = k \qquad \Longrightarrow \qquad H = P_0 + k\vec{\mathbf{d}} = Q.$$

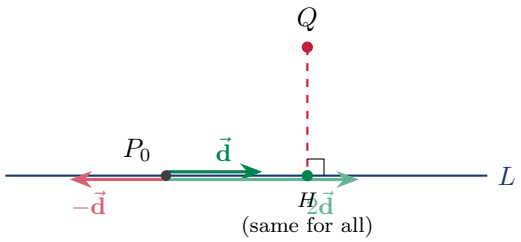
(Division is legitimate because $\vec{\mathbf{d}} \cdot \vec{\mathbf{d}} = \|\vec{\mathbf{d}}\|^2 > 0$.) Geometrically: the closest point of L to a point of L is the point itself, at distance 0. Consistently, $\overrightarrow{HQ} = \vec{\mathbf{0}}$, which is (trivially) perpendicular to $\vec{\mathbf{d}}$.

Problem 13 ⋆⋆ EXT. Does the choice of $\vec{\mathbf{d}}$ matter?

With $\vec{\mathbf{d}}' = k\vec{\mathbf{d}}$:

$$t' = \frac{\vec{\mathbf{w}} \cdot (k\vec{\mathbf{d}})}{(k\vec{\mathbf{d}}) \cdot (k\vec{\mathbf{d}})} = \frac{k\,(\vec{\mathbf{w}} \cdot \vec{\mathbf{d}})}{k^2\,(\vec{\mathbf{d}} \cdot \vec{\mathbf{d}})} = \frac{t}{k}, \qquad H' = P_0 + t'\vec{\mathbf{d}}' = P_0 + \frac{t}{k}\,(k\vec{\mathbf{d}}) = P_0 + t\vec{\mathbf{d}} = H.$$

The *parameter* changes ($t' = t/k$), but the *point* does not.
Why expected. $\vec{\mathbf{d}}$ and $k\vec{\mathbf{d}}$ describe the same line L (same set of points). The projection H is defined geometrically, as the point of L with $QH \perp L$; it depends on the line, not on how we parametrize it. Changing $\vec{\mathbf{d}}$ to $k\vec{\mathbf{d}}$ only changes the "speed" (and, if $k < 0$, the direction of travel) along L.



Distance from a Point to a Line

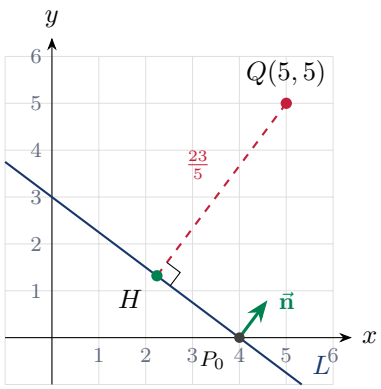
Problem 14 ⋆ Cartesian shortcut

With $A=3$, $B=4$, $C=-12$ and $(x_0,y_0)=(5,5)$:

$$d(Q,L)=\frac{|3(5)+4(5)-12|}{\sqrt{3^2+4^2}}=\frac{|23|}{5}=\frac{23}{5}=4.6.$$

Where the formula comes from. Take any point of L, e.g. $P_0=(4,0)$. The distance is the length of the component of $Q-P_0=(1,5)$ along the unit normal $\vec{\mathbf{n}}/\|\vec{\mathbf{n}}\|$:

$$\frac{|\vec{\mathbf{n}}\cdot(Q-P_0)|}{\|\vec{\mathbf{n}}\|}=\frac{|3+20|}{5}=\frac{23}{5}.$$



Check. the foot is $H=Q-\frac{23}{25}(3,4)=\left(\frac{56}{25},\frac{33}{25}\right)$: $3\cdot\frac{56}{25}+4\cdot\frac{33}{25}-12=\frac{168+132-300}{25}=0$ ✓.

Problem 15 ⋆⋆ Two methods

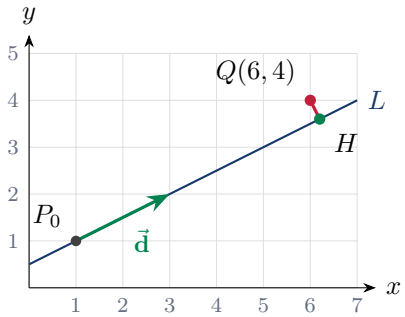
(a) $\vec{\mathbf{w}}=Q-P_0=(5,3)$, $\vec{\mathbf{w}}\cdot\vec{\mathbf{d}}=10+3=13$, $\vec{\mathbf{d}}\cdot\vec{\mathbf{d}}=5$, $t=\frac{13}{5}$.

$$H=(1,1)+\tfrac{13}{5}(2,1)=\left(\tfrac{31}{5},\ \tfrac{18}{5}\right),$$

$$Q-H=\left(-\tfrac{1}{5},\ \tfrac{2}{5}\right),\quad d(Q,L)=\tfrac{1}{5}\sqrt{1+4}=\tfrac{\sqrt{5}}{5}\approx 0.447.$$

(b) $\vec{\mathbf{d}}=(2,1)\Rightarrow\vec{\mathbf{n}}=(1,-2)$: $1(x-1)-2(y-1)=0\iff x-2y+1=0$.

$$d(Q,L)=\frac{|6-8+1|}{\sqrt{1+4}}=\frac{1}{\sqrt{5}}=\frac{\sqrt{5}}{5}\ \checkmark$$



Q is very close to L: $d\approx 0.45$.

(c) Once the Cartesian equation is known, the shortcut is a one-line computation, so it was quicker here (even counting the time to find the equation). In $\mathbb{R}^3$ the shortcut does not exist (a single equation

$Ax + By + Cz + D = 0$ is a plane), so we must use projection or the cross product.

Problem 16 ⋆⋆ Projection in space

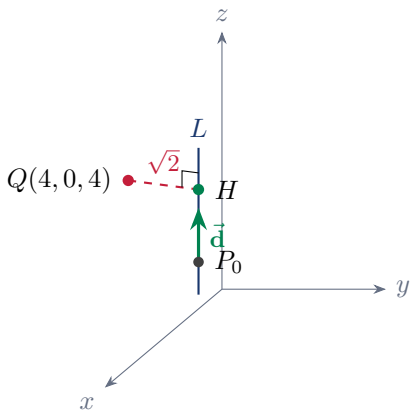
(a) $\vec{\mathbf{w}} = Q - P_0 = (3, 0, 3)$, $\vec{\mathbf{w}} \cdot \vec{\mathbf{d}} = 6 + 0 + 6 = 12$, $\vec{\mathbf{d}} \cdot \vec{\mathbf{d}} = 4 + 1 + 4 = 9$, $t = \frac{12}{9} = \frac{4}{3}$.

$$H = (1, 0, 1) + \tfrac{4}{3}(2, 1, 2) = \left(\tfrac{11}{3},\ \tfrac{4}{3},\ \tfrac{11}{3}\right).$$

(b) $\overrightarrow{HQ} = Q - H = \left(\tfrac{1}{3},\ -\tfrac{4}{3},\ \tfrac{1}{3}\right)$,

$$d(Q, L) = \tfrac{1}{3}\sqrt{1 + 16 + 1} = \tfrac{\sqrt{18}}{3} = \sqrt{2} \approx 1.41.$$

(c) $\overrightarrow{HQ} \cdot \vec{\mathbf{d}} = \frac{2}{3} - \frac{4}{3} + \frac{2}{3} = 0$ ✓.



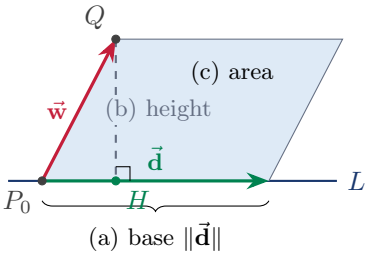
Problem 17 ⋆⋆ Cross-product shortcut

(a) $\vec{\mathbf{w}} = (3, 0, 3)$ and

$$\vec{\mathbf{d}} \times \vec{\mathbf{w}} = (2, 1, 2) \times (3, 0, 3) = \big(1 \cdot 3 - 2 \cdot 0,\ 2 \cdot 3 - 2 \cdot 3,\ 2 \cdot 0 - 1 \cdot 3\big) = (3, 0, -3),$$

so $\|\vec{\mathbf{d}} \times \vec{\mathbf{w}}\| = \sqrt{18} = 3\sqrt{2}$, $\|\vec{\mathbf{d}}\| = 3$ and $d(Q, L) = \dfrac{3\sqrt{2}}{3} = \sqrt{2}$, the same as in Problem 16.

(b) (a) is along L, under the side $\vec{\mathbf{d}}$: it is the base $\|\vec{\mathbf{d}}\|$.
(b) labels the dashed segment QH: it is the height $d(Q, L)$.
(c) sits inside the parallelogram: it is its area $\|\vec{\mathbf{d}} \times \vec{\mathbf{w}}\|$.



(c) The parallelogram built on $\vec{\mathbf{d}}$ and $\vec{\mathbf{w}}$ has

$$\text{area} = \|\vec{\mathbf{d}} \times \vec{\mathbf{w}}\|,$$
$$\text{area} = \text{base} \times \text{height} = \|\vec{\mathbf{d}}\| \cdot QH.$$

Equating, $QH = d(Q, L) = \dfrac{\|\vec{\mathbf{d}} \times \vec{\mathbf{w}}\|}{\|\vec{\mathbf{d}}\|}$.

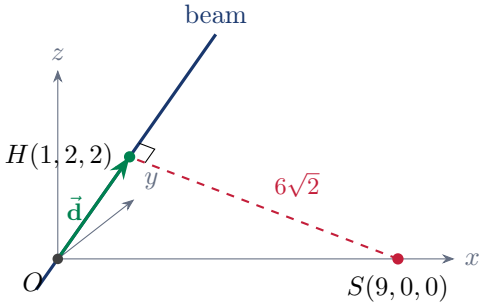
Problem 18 ⋆⋆ Applied

$P_0 = O$, $\vec{\mathbf{d}} = (1,2,2)$, $Q = S = (9,0,0)$, $\vec{\mathbf{w}} = (9,0,0)$.

(b) (done first, since (a) needs it) $\vec{\mathbf{w}} \cdot \vec{\mathbf{d}} = 9$, $\vec{\mathbf{d}} \cdot \vec{\mathbf{d}} = 1+4+4 = 9$, so $t = 1$ and the closest point of the beam is $H = (1,2,2)$.

(a) $S - H = (8,-2,-2)$ and

$$d_{\min} = \sqrt{64+4+4} = \sqrt{72} = 6\sqrt{2} \approx 8.49.$$



Check. $(8,-2,-2)\cdot(1,2,2) = 8-4-4 = 0$ ✓.
Cross product: $\vec{\mathbf{d}} \times \vec{\mathbf{w}} = (0,18,-18)$, $\frac{18\sqrt{2}}{3} = 6\sqrt{2}$ ✓.

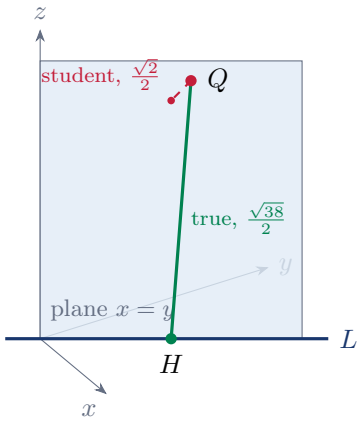
Problem 19 ⋆⋆ EXT. Find the error

(a) In $\mathbb{R}^3$, the equation $x - y = 0$ places no condition on z: it is the **vertical plane** $\{x = y\}$, which contains the given line *and* the z-axis. The student applied the $\mathbb{R}^2$ formula, which in $\mathbb{R}^3$ computes the distance from Q to that *plane*: $\frac{|1-2|}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx 0.71$. The plane contains the line, so this number is *smaller* than the true distance.

(b) $\vec{\mathbf{w}} = (1,2,3)$, $\vec{\mathbf{d}} = (1,1,0)$, $\vec{\mathbf{w}} \cdot \vec{\mathbf{d}} = 3$, $\vec{\mathbf{d}} \cdot \vec{\mathbf{d}} = 2$, $t = \frac{3}{2}$:

$$H = \left(\tfrac{3}{2}, \tfrac{3}{2}, 0\right), \quad Q - H = \left(-\tfrac{1}{2}, \tfrac{1}{2}, 3\right),$$

$$d(Q, L) = \sqrt{\tfrac{1}{4} + \tfrac{1}{4} + 9} = \sqrt{\tfrac{19}{2}} = \tfrac{\sqrt{38}}{2} \approx 3.08.$$

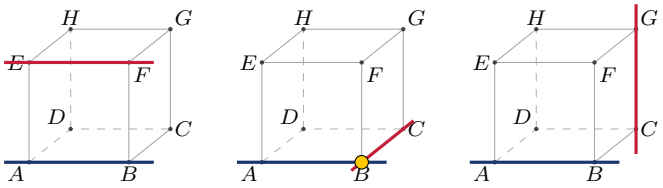


Check. $\vec{\mathbf{d}} \times \vec{\mathbf{w}} = (3,-3,1)$, $\frac{\sqrt{19}}{\sqrt{2}} = \frac{\sqrt{38}}{2}$ ✓.

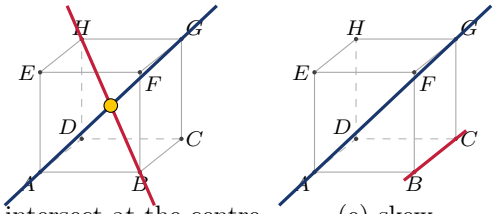
Relative Position of Two Lines

Problem 20 ⋆⋆ Lines on a cube

In every case: (PQ) is written $P + t\,\overrightarrow{PQ}$ (parameter t for the first line, s for the second).



(a) parallel distinct (b) intersect at B (c) skew



(d) intersect at the centre (e) skew

(a) (AB): $(t, 0, 0)$, $\vec{\mathbf{u}} = (1, 0, 0)$. (EF): $(s, 0, 1)$, $\vec{\mathbf{v}} = (1, 0, 0)$.
Prediction: opposite edges of the front face, parallel. *Test:* $\vec{\mathbf{v}} = \vec{\mathbf{u}}$, so parallel. Is $E(0, 0, 1)$ on (AB)? Every point of (AB) has $z = 0 \neq 1$. **Parallel and distinct.**

(b) (AB): $(t, 0, 0)$. (BC): $(1, s, 0)$, $\vec{\mathbf{v}} = (0, 1, 0)$, not a multiple of $\vec{\mathbf{u}}$.
Prediction: they share the vertex B. *Test:* x: $t = 1$; y: $0 = s$; z: $0 = 0$ ✓. **Intersect** at $(1, 0, 0) = B$.

(c) (AB): $(t, 0, 0)$. (CG): $(1, 1, s)$, $\vec{\mathbf{v}} = (0, 0, 1)$, not parallel to $\vec{\mathbf{u}}$.
Prediction: (AB) is a bottom edge, (CG) a vertical edge at the back that does not touch it: skew. *Test:* x: $t = 1$; z: $0 = s$; then y: $0 = 1$ **false**. Not parallel and no common point: **skew**.

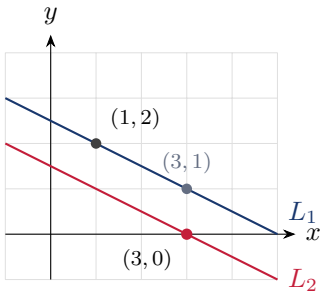
(d) (AG): (t, t, t), $\vec{\mathbf{u}} = (1, 1, 1)$. (BH): $B + s(H - B) = (1 - s,\ s,\ s)$, $\vec{\mathbf{v}} = (-1, 1, 1)$, not parallel.
Prediction: two space diagonals cross at the centre of the cube. *Test:* x: $t = 1 - s$; y: $t = s$. Substituting, $s = 1 - s$, so $s = \frac{1}{2}$ and $t = \frac{1}{2}$. z: $t = s$, i.e. $\frac{1}{2} = \frac{1}{2}$ ✓. **Intersect** at $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.

(e) (AG): (t, t, t). (BC): $(1, s, 0)$, not parallel.
Test: x: $t = 1$; y: $s = t = 1$; z: $t = 0$, i.e. $1 = 0$ **false**. **Skew.** The diagonal (AG) passes "above" the edge (BC): at $x = 1$ it is already at height $z = 1$.

Problem 21 ⋆ Parallel: equal or distinct?

(a) $(-4,2) = -2\,(2,-1)$: the directions are parallel. Is $(3,0) \in L_1$? L_1: $x = 1+2t$, $y = 2-t$. From x: $1+2t = 3 \Rightarrow t = 1$; then $y = 2-1 = 1 \neq 0$. **Parallel and distinct.**

(b) $\vec{\mathbf{v}} = 2\vec{\mathbf{u}}$ (equivalently $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \vec{\mathbf{0}}$): parallel. Is $(0,1,1) \in L_1$? L_1: $(1+t,\ t,\ 0)$ has $z = 0 \neq 1$ for every t. **Parallel and distinct.**



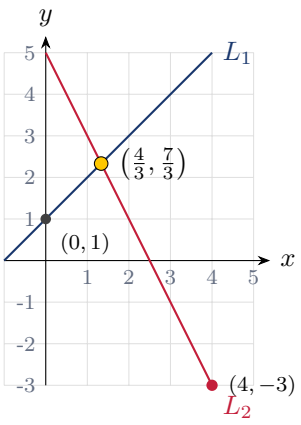
(a): L_1 passes through $(3,1)$, not $(3,0)$.

Problem 22 ⋆ Intersection in the plane

L_1: $x = t,\ y = 1+t$. L_2: $x = 4-s,\ y = -3+2s$.
Directions $(1,1)$ and $(-1,2)$ are not parallel, so the lines meet at exactly one point.
x: $t = 4-s$. Substituting in y: $1+(4-s) = -3+2s \Rightarrow 8 = 3s \Rightarrow s = \frac{8}{3}$, then $t = 4-\frac{8}{3} = \frac{4}{3}$.

$$L_1\left(t = \tfrac{4}{3}\right) = \left(\tfrac{4}{3},\ \tfrac{7}{3}\right).$$

Check. $L_2\left(s = \frac{8}{3}\right) = \left(4-\frac{8}{3},\ -3+\frac{16}{3}\right) = \left(\frac{4}{3},\ \frac{7}{3}\right)$ ✓.



Problem 23 ⋆⋆ Skew test

L_1: $(t,\ 0,\ t)$. L_2: $(1,\ 1+s,\ s)$.
Parallel? $(1,0,1)$ and $(0,1,1)$ are not multiples of each other (the first has y-component 0, the second does not), so no.
Common point? x: $t = 1$. y: $0 = 1+s \Rightarrow s = -1$. **Test** z: $t = s$ would require $1 = -1$, **false**.
Not parallel and no common point: the lines are **skew**.

Problem 24 ⋆⋆ Intersection in space

L_1: $(t,\ t,\ t)$. L_2: $(3-2s,\ -1+2s,\ 2-s)$.
Parallel? $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (1,1,1) \times (-2,2,-1) = \big(1(-1)-1(2),\ 1(-2)-1(-1),\ 1(2)-1(-2)\big) = (-3,-1,4) \neq \vec{\mathbf{0}}$. Not parallel.

(a) x: $t = 3 - 2s$. y: $t = -1 + 2s$. Substituting: $3 - 2s = -1 + 2s \Rightarrow s = 1$, then $t = 1$.
Test z: $t = 2 - s$ gives $1 = 2 - 1$ ✓. The lines **intersect** at $L_1(1) = (1, 1, 1)$.

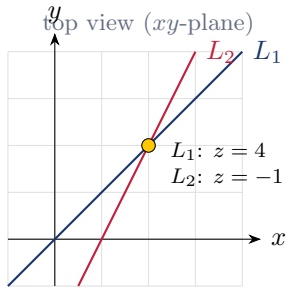
Check. $L_2(1) = (3 - 2,\ -1 + 2,\ 2 - 1) = (1, 1, 1)$ ✓.

(b) y: $t = -1 + 2s$. z: $t = 2 - s$. Substituting: $-1 + 2s = 2 - s \Rightarrow s = 1$, $t = 1$. **Test** x: $3 - 2(1) = 1 = t$ ✓. Same conclusion, same point.
Why it must agree. "The lines have a common point" means there exist t, s satisfying *all three* equations at once. That is a fact about the lines, not about the order in which we write the equations. If such (t, s) exists, any two independent equations will find it and the third will confirm it; if it does not exist, whatever pair we solve, the remaining equation must fail.

Problem 25 ⋆⋆ EXT. Find the error

The student solved x: $t = 1 + s$ and y: $t = 2s$ (giving $s = 1$, $t = 2$) but **never tested the z-equation**: $2t = -s$ would require $4 = -1$, **false**.
Indeed $L_1(2) = (2, 2, 4)$ but $L_2(1) = (2, 2, -1)$: same x and y, different heights. One point is directly above the other.
The directions $(1, 1, 2)$ and $(1, 2, -1)$ are not parallel, and there is no common point: the lines are **skew**.

Remark. Solving only x and y amounts to looking at the lines *from above* (projecting onto the xy-plane). In that view they cross at $(2, 2)$, exactly like skew lines that "look" as if they cross in a drawing.

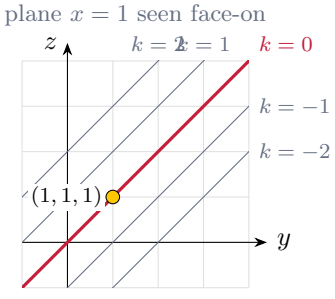


Problem 26 ⋆⋆⋆ EXT. A parameter

$\vec{\mathbf{u}} = (1, 1, 1)$, $\vec{\mathbf{v}} = (0, 1, 1)$: $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (1 \cdot 1 - 1 \cdot 1,\ 1 \cdot 0 - 1 \cdot 1,\ 1 \cdot 1 - 1 \cdot 0) = (0, -1, 1) \neq \vec{\mathbf{0}}$. Never parallel, whatever k.
x: $t = 1$. y: $t = s \Rightarrow s = 1$. **Test** z: $t = k + s$, i.e. $1 = k + 1 \iff k = 0$.

- $k = 0$: the lines **intersect** at $(1, 1, 1)$.
- $k \neq 0$: not parallel and no common point: **skew**.

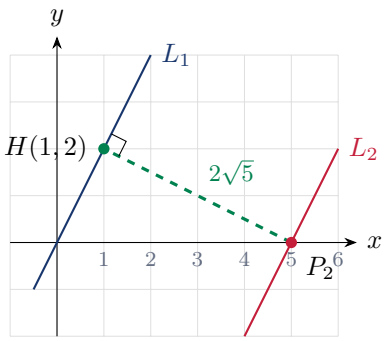
Geometric reading. For every k, L_2 lies in the plane $x = 1$, and changing k slides L_2 up or down inside that plane. L_1 crosses the plane $x = 1$ at a single point, $(1, 1, 1)$. So L_2 meets L_1 only for the one position that passes through $(1, 1, 1)$.



Distance Between Two Lines

Problem 27 ⋆ Parallel lines in the plane

The lines are parallel, so $d(L_1, L_2) = d(P_2, L_1)$ for any point $P_2 \in L_2$; take $P_2 = (5, 0)$.



(a) $\vec{\mathbf{w}} = P_2 - P_1 = (5, 0)$, $\vec{\mathbf{d}} = (1, 2)$, $\vec{\mathbf{w}} \cdot \vec{\mathbf{d}} = 5$, $\vec{\mathbf{d}} \cdot \vec{\mathbf{d}} = 5$, $t = 1$:

$$H = (1, 2), \quad P_2 - H = (4, -2),$$
$$d = \sqrt{16 + 4} = 2\sqrt{5} \approx 4.47.$$

Check. $(4, -2) \cdot (1, 2) = 0$ ✓.

(b) L_1: normal $(2, -1)$, through the origin: $2x - y = 0$.

$$d(P_2, L_1) = \frac{|2(5) - 0|}{\sqrt{4 + 1}} = \frac{10}{\sqrt{5}} = 2\sqrt{5}$$ ✓

Problem 28 ⋆⋆ Distances on the cube

Coordinates as in Problem 20.

(a) *Guesses:* (AB) and (EF) are opposite edges of a face of side 1: distance 1. (AB) and (CG) are skew, and the edge $[BC]$ is perpendicular to both: distance 1 again.

(AB) and (EF) (parallel): $d = d(E, (AB))$. With $P_0 = A$, $\vec{\mathbf{d}} = (1, 0, 0)$, $\vec{\mathbf{w}} = E - A = (0, 0, 1)$: $\vec{\mathbf{w}} \cdot \vec{\mathbf{d}} = 0$, so $t = 0$, $H = A$, and $d = \|E - A\| = 1$. The common perpendicular is the edge $[AE]$.

(AB) and (CG) (skew, Problem 20(c)): $\vec{\mathbf{u}} = (1, 0, 0)$, $\vec{\mathbf{v}} = (0, 0, 1)$,

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (0 \cdot 1 - 0 \cdot 0,\ 0 \cdot 0 - 1 \cdot 1,\ 1 \cdot 0 - 0 \cdot 0) = (0, -1, 0),$$

$$d = \frac{|(C - A) \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}})|}{\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|} = \frac{|(1, 1, 0) \cdot (0, -1, 0)|}{1} = 1.$$

The common perpendicular is the edge $[BC]$: $\overrightarrow{BC} = (0, 1, 0)$ is perpendicular to $(1, 0, 0)$ and to $(0, 0, 1)$.

(b) (AG): $P_1 = A$, $\vec{\mathbf{u}} = (1, 1, 1)$. (BC): $P_2 = B$, $\vec{\mathbf{v}} = (0, 1, 0)$. They are skew (Problem 20(e)).

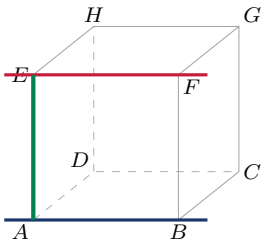
$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (1 \cdot 0 - 1 \cdot 1,\ 1 \cdot 0 - 1 \cdot 0,\ 1 \cdot 1 - 1 \cdot 0) = (-1, 0, 1), \qquad d = \frac{|(1, 0, 0) \cdot (-1, 0, 1)|}{\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx 0.71.$$

Where is the shortest bridge? Normal equations with $P_1 - P_2 = (-1, 0, 0)$, $\vec{\mathbf{u}} \cdot \vec{\mathbf{u}} = 3$, $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 1$, $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = 1$:

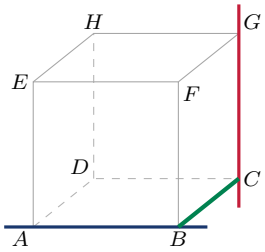
$$-1 + 3t - s = 0, \qquad 0 + t - s = 0.$$

The second gives $s = t$; substituting, $-1 + 2t = 0$, so $t = s = \frac{1}{2}$. Then $H_1 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is the **centre of the cube** and $H_2 = \left(1, \frac{1}{2}, 0\right)$ is the **midpoint of** $[BC]$; $\|H_2 - H_1\| = \left\|\left(\frac{1}{2}, 0, -\frac{1}{2}\right)\right\| = \frac{\sqrt{2}}{2}$ ✓.

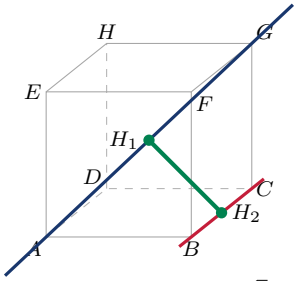
Expected? Many students guess 1 ("an edge length") or something involving $\sqrt{3}$ (the diagonal). Neither: $\overrightarrow{H_1H_2} = \left(\frac{1}{2}, 0, -\frac{1}{2}\right) = \frac{1}{2}\overrightarrow{EB}$: the bridge is parallel to the face diagonal $[EB]$ and is exactly half of it, which is why $\frac{\sqrt{2}}{2}$ appears.



(AB), (EF): bridge $[AE]$, $d = 1$



(AB), (CG): bridge $[BC]$, $d = 1$



(AG), (BC): $d = \frac{\sqrt{2}}{2}$

Problem 29 ⋆⋆ A genuine skew pair

L_1: $(t,\ t,\ 0)$. $\quad L_2$: $(1,\ s,\ 1+s)$.

(a) *Parallel?* $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (1,1,0) \times (0,1,1) = (1\cdot 1 - 0\cdot 1,\ 0\cdot 0 - 1\cdot 1,\ 1\cdot 1 - 1\cdot 0) = (1,-1,1) \neq \vec{\mathbf{0}}$: no. *Common point?* x: $t = 1$. y: $t = s \Rightarrow s = 1$. **Test** z: $0 = 1 + s = 2$, **false**. The lines are **skew**.

(b) $P_1 - P_2 = (0,0,0) - (1,0,1) = (-1,0,-1)$, so $(P_1 - P_2)\cdot\vec{\mathbf{u}} = -1$, $(P_1 - P_2)\cdot\vec{\mathbf{v}} = -1$; and $\vec{\mathbf{u}}\cdot\vec{\mathbf{u}} = 2$, $\vec{\mathbf{u}}\cdot\vec{\mathbf{v}} = 1$, $\vec{\mathbf{v}}\cdot\vec{\mathbf{v}} = 2$.

$$\text{(N1)} \quad -1 + 2t - s = 0, \qquad \text{(N2)} \quad -1 + t - 2s = 0.$$

From (N1): $s = 2t - 1$. Substituting in (N2): $-1 + t - 2(2t-1) = 0 \iff 1 - 3t = 0 \iff t = \frac{1}{3}$, then $s = \frac{2}{3} - 1 = -\frac{1}{3}$.

(c) $H_1 = L_1\left(\frac{1}{3}\right) = \left(\frac{1}{3}, \frac{1}{3}, 0\right)$, $H_2 = L_2\left(-\frac{1}{3}\right) = \left(1, -\frac{1}{3}, \frac{2}{3}\right)$.

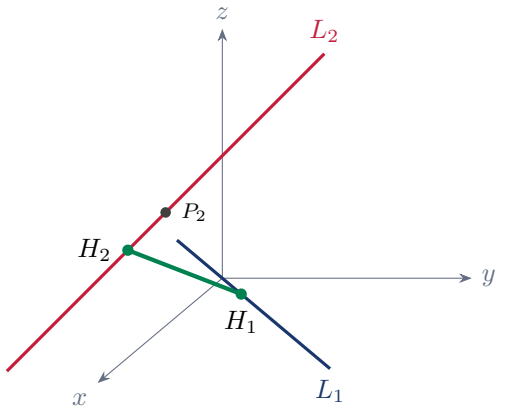
$$\overrightarrow{H_1H_2} = \left(\tfrac{2}{3}, -\tfrac{2}{3}, \tfrac{2}{3}\right),$$

$$d(L_1, L_2) = \tfrac{2}{3}\sqrt{3} = \tfrac{2}{\sqrt{3}} = \tfrac{2\sqrt{3}}{3} \approx 1.15.$$

(d) $\overrightarrow{H_1H_2}\cdot\vec{\mathbf{u}} = \frac{2}{3} - \frac{2}{3} + 0 = 0$ ✓ $\quad \overrightarrow{H_1H_2}\cdot\vec{\mathbf{v}} = 0 - \frac{2}{3} + \frac{2}{3} = 0$ ✓

Note $\overrightarrow{H_1H_2} = \frac{2}{3}(1,-1,1) = \frac{2}{3}\,(\vec{\mathbf{u}} \times \vec{\mathbf{v}})$, as it must be.

(e) $\dfrac{|(P_2 - P_1)\cdot(\vec{\mathbf{u}} \times \vec{\mathbf{v}})|}{\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|} = \dfrac{|(1,0,1)\cdot(1,-1,1)|}{\sqrt{3}} = \dfrac{2}{\sqrt{3}}$ ✓



L_1 lies in the xy-plane; L_2 lies in the plane $x = 1$.

Problem 30 ⋆⋆⋆ EXT. The one-variable route

$P_t = (t, t, 0)$, $P_2 = (1, 0, 1)$, $\vec{\mathbf{v}} = (0, 1, 1)$.

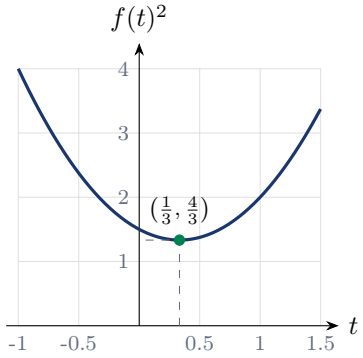
(a) Projecting P_t on L_2 with the usual recipe:

$$s(t) = \frac{(P_t - P_2)\cdot \vec{\mathbf{v}}}{\vec{\mathbf{v}}\cdot\vec{\mathbf{v}}} = \frac{(t-1,\ t,\ -1)\cdot(0,1,1)}{2} = \frac{t-1}{2}.$$

It is an **affine** function of t (degree 1). Hence $Q_t = \left(1,\ \frac{t-1}{2},\ \frac{t+1}{2}\right)$.

(b) $P_t - Q_t = \left(t-1,\ t-\frac{t-1}{2},\ -\frac{t+1}{2}\right) = \left(t-1,\ \frac{t+1}{2},\ -\frac{t+1}{2}\right)$, so

$$f(t)^2 = (t-1)^2 + 2\cdot\frac{(t+1)^2}{4}$$
$$= t^2 - 2t + 1 + \frac{t^2+2t+1}{2} = \frac{3}{2}t^2 - t + \frac{3}{2}.$$



(c) This is an upward parabola. $\frac{d}{dt}(f^2) = 3t - 1 = 0 \iff t^* = \frac{1}{3}$ (equivalently, the vertex is at $-\frac{b}{2a} = \frac{1}{3}$), and

$$f(t^*)^2 = \tfrac{3}{2}\cdot\tfrac{1}{9} - \tfrac{1}{3} + \tfrac{3}{2} = \tfrac{1-2+9}{6} = \tfrac{4}{3}, \quad \min f = \tfrac{2}{\sqrt{3}}.$$

Same as Problem 29: $t^* = \frac{1}{3}$ and $s(t^*) = -\frac{1}{3}$ are the parameters of H_1, H_2, and the minimum is the distance $\frac{2}{\sqrt{3}}$.

(d) $(P_t - Q_t)\cdot\vec{\mathbf{v}} = 0 + \frac{t+1}{2} - \frac{t+1}{2} = 0$ for *every* t: this is built in, because Q_t was *defined* as the orthogonal projection of P_t on L_2.
At $t^* = \frac{1}{3}$: $P_{t^*} - Q_{t^*} = \left(-\frac{2}{3}, \frac{2}{3}, -\frac{2}{3}\right)$ and $\left(-\frac{2}{3}, \frac{2}{3}, -\frac{2}{3}\right)\cdot(1,1,0) = 0$ ✓. This second fact holds *only* at t^*, and it is the one that comes from setting the derivative to zero: moving P_t along L_1 no longer changes the distance (to first order) exactly when the connecting segment is perpendicular to L_1.

Problem 31 ⋆⋆⋆ EXT. A harder skew pair

L_1: $P_1 = (0,0,0)$, $\vec{\mathbf{u}} = (1,1,2)$. L_2: $P_2 = (1,0,0)$, $\vec{\mathbf{v}} = (1,2,-1)$. They are skew (Problem 25).
Cross-product method.

$$\vec{\mathbf{u}}\times\vec{\mathbf{v}} = \left(1(-1) - 2\cdot 2,\ 2\cdot 1 - 1(-1),\ 1\cdot 2 - 1\cdot 1\right) = (-5, 3, 1), \qquad \|\vec{\mathbf{u}}\times\vec{\mathbf{v}}\| = \sqrt{35},$$

$$d(L_1, L_2) = \frac{|(1,0,0)\cdot(-5,3,1)|}{\sqrt{35}} = \frac{5}{\sqrt{35}} = \frac{\sqrt{35}}{7} \approx 0.845.$$

Check with the normal equations. $P_1 - P_2 = (-1,0,0)$: $(P_1-P_2)\cdot\vec{\mathbf{u}} = -1$, $(P_1-P_2)\cdot\vec{\mathbf{v}} = -1$; $\vec{\mathbf{u}}\cdot\vec{\mathbf{u}} = 6$, $\vec{\mathbf{u}}\cdot\vec{\mathbf{v}} = 1 + 2 - 2 = 1$, $\vec{\mathbf{v}}\cdot\vec{\mathbf{v}} = 6$.

$$-1 + 6t - s = 0, \qquad -1 + t - 6s = 0.$$

First: $s = 6t - 1$. Second: $-1 + t - 36t + 6 = 0 \iff t = \frac{1}{7}$, then $s = -\frac{1}{7}$.
$H_1 = \frac{1}{7}(1, 1, 2)$, $H_2 = \left(\frac{6}{7}, -\frac{2}{7}, \frac{1}{7}\right)$, $\overrightarrow{H_1H_2} = \frac{1}{7}(5, -3, -1)$, of length $\frac{\sqrt{35}}{7}$ ✓, and parallel to $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ as expected.

Synthesis

Problem 32 ⋆⋆ Two lines, one decision

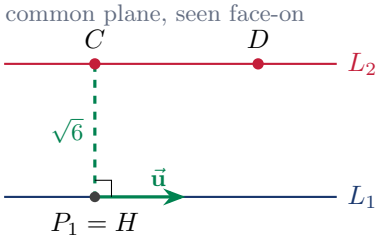
(a) $\overrightarrow{CD} = (2, -2, 2) = 2(1, -1, 1)$; take $\vec{\mathbf{v}} = (1, -1, 1)$.

(b) $\vec{\mathbf{v}} = \vec{\mathbf{u}}$: the lines are **parallel**. Is $C(0, 3, 2)$ a point $(1 + t,\ 2 - t,\ t)$ of L_1?
x: $1 + t = 0 \Rightarrow t = -1$; y: $2 - (-1) = 3$ ✓; z: $t = -1 \neq 2$ **fails**.
Testing all three coordinates matters: x and y alone would wrongly suggest $C \in L_1$. **Parallel and distinct.**

(c) The relevant quantity is the distance $d(L_1, L_2) = d(C, L_1)$. $\vec{\mathbf{w}} = C - P_1 = (-1, 1, 2)$, $\vec{\mathbf{w}} \cdot \vec{\mathbf{u}} = -1 - 1 + 2 = 0$, so $t = 0$ and $H = P_1 = (1, 2, 0)$:

$$d = \|C - P_1\| = \sqrt{1 + 1 + 4} = \sqrt{6} \approx 2.45.$$

Check. $\vec{\mathbf{u}} \times \vec{\mathbf{w}} = (-3, -3, 0)$, $\frac{3\sqrt{2}}{\sqrt{3}} = \sqrt{6}$ ✓.



$\vec{\mathbf{w}} \perp \vec{\mathbf{u}}$: C projects exactly onto P_1.

Problem 33 ⋆⋆ A triangle in space

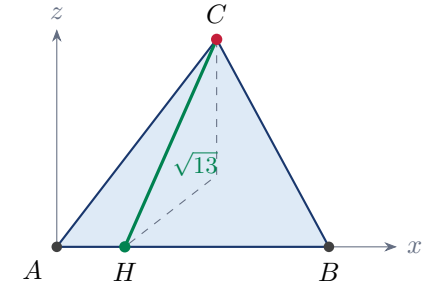
(a) $\overrightarrow{AB} = (4,0,0)$: $(x,y,z) = t(4,0,0)$, i.e. $(4t,0,0)$ (the x-axis).

(b) $\vec{\mathbf{w}} = C - A = (1,3,2)$, $\vec{\mathbf{w}} \cdot \vec{\mathbf{d}} = 4$, $\vec{\mathbf{d}} \cdot \vec{\mathbf{d}} = 16$, $t = \frac{1}{4}$, $H = (1,0,0)$.

Check. $\overrightarrow{HC} = (0,3,2)$, $(0,3,2) \cdot (4,0,0) = 0$ ✓.

(c) Height $\|C - H\| = \sqrt{9+4} = \sqrt{13}$, base $AB = 4$:

$$\text{Area} = \tfrac{1}{2} \cdot 4 \cdot \sqrt{13} = 2\sqrt{13} \approx 7.21.$$



$\overrightarrow{HC} = (0,3,2)$ has no x-component: it is perpendicular to the x-axis.

(d) $\overrightarrow{AB} \times \overrightarrow{AC} = (4,0,0) \times (1,3,2)$

$= (0 \cdot 2 - 0 \cdot 3,\ 0 \cdot 1 - 4 \cdot 2,\ 4 \cdot 3 - 0 \cdot 1) = (0,-8,12)$,

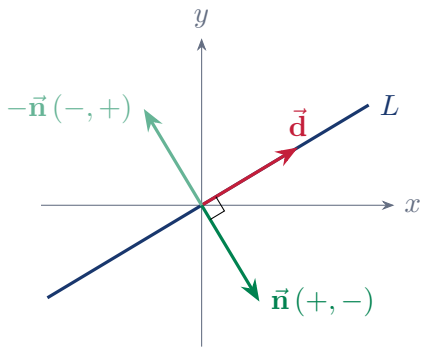
$\|(0,-8,12)\| = \sqrt{208} = 4\sqrt{13}$, and $\frac{1}{2} \cdot 4\sqrt{13} = 2\sqrt{13}$ ✓.

Problem 34 ⋆ Which point is the projection?

The answer is $\boxed{B}$. The orthogonal projection H of Q on L is characterized by a single property: $\overrightarrow{HQ} \perp L$. In the figure only the segment QB makes a right angle with L.

Why this point is also the closest. For any other point $X \in L$, the triangle QBX has a right angle at B, so by Pythagoras $QX^2 = QB^2 + BX^2 > QB^2$. Hence $QA > QB$ and $QC > QB$: the oblique segments are always longer.

Problem 35 ⋆⋆ Read the geometry before writing equations



(a) Moving along $\vec{\mathbf{d}}$ we go *right* and *up*: $a > 0$ and $b > 0$.

(b) $\vec{\mathbf{n}}$ is $\vec{\mathbf{d}}$ turned by a quarter turn. Turning a "right-and-up" arrow by 90° gives either a "right-and-down" arrow or a "left-and-up" arrow. So the signs of (A, B) are $(+,-)$ or $(-,+)$: **opposite signs**. (Algebraically, $\vec{\mathbf{n}} = \pm(b, -a)$.)

(c) The coefficients (A, B) of $Ax + By + C = 0$ form a normal vector. If $A > 0$, the normal points to the right, and the only normal pattern with $A > 0$ is $(+,-)$: so $\boldsymbol{B < 0}$. Equivalently, $\vec{\mathbf{n}} \cdot \vec{\mathbf{d}} = Aa + Bb = 0$ with $a, b > 0$: since $Aa > 0$, the term Bb must be negative to cancel it, so $B < 0$.

Problem 36 ⋆⋆ The shortest bridge between skew lines

(a) The segment $[H_1H_2]$: it is the common perpendicular, and its length is $d(L_1, L_2)$.

(b) $\overrightarrow{H_1H_2} \cdot \vec{\mathbf{u}} = 0$ and $\overrightarrow{H_1H_2} \cdot \vec{\mathbf{v}} = 0$.

(c) P and Q are arbitrary points of L_1 and L_2. Nothing forces PQ to be perpendicular to either line, and if it is not perpendicular to L_1, say, we can slide P along L_1 and get closer (as in Problems 30 and 34). In general $PQ \geq H_1H_2$, with equality only for $P = H_1$, $Q = H_2$.

(d) $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is perpendicular to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, so it gives the *direction* of the common perpendicular $\overrightarrow{H_1H_2}$. Writing $P_2 - P_1 = \overrightarrow{P_1H_1} + \overrightarrow{H_1H_2} + \overrightarrow{H_2P_2}$, the first and last vectors are along $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, hence perpendicular to $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$. So

$$(P_2 - P_1) \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = \overrightarrow{H_1H_2} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}}),$$

and dividing by $\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|$ gives $\pm\|\overrightarrow{H_1H_2}\|$: the formula measures the component of *any* connecting vector $P_2 - P_1$ along the common-normal direction. That is why any two points P_1, P_2 can be used.

Problem 37 ⋆ True or false

(a) **False.** In $\mathbb{R}^3$, $Ax + By + Cz + D = 0$ describes a plane; a line needs two independent such equations.

(b) **False.** They are parallel, but may be distinct: e.g. (AB) and (EF) on the cube (Problem 20(a)).

(c) **False.** Skew lines need a third dimension; in $\mathbb{R}^2$ two non-parallel lines always meet at exactly one point.

(d) **True.** If $Q = P_0 + k\vec{\mathbf{d}}$, the formula gives $t = k$ and $H = Q$ (Problem 12).

(e) True. A common point gives two points (one on each line) at distance 0, and a distance cannot be negative.

(f) False. In $\mathbb{R}^3$ non-parallel lines may be skew, e.g. Problems 23, 25 and 29.