Summary of Chapter 4

Planes in Space — definitions, projection, distance and intersections

Mohamed Nouidha
Collège Sainte-Anne

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1. Context

Key idea

A line needs one direction; a plane needs **two independent directions**. Rather than carrying two directions around, we summarize them with a single vector perpendicular to both: the **normal** $\vec{\mathbf{n}}$. Almost everything in this chapter (equations, projection, distance, intersections) comes down to dot products with $\vec{\mathbf{n}}$.

Reminder

$\vec{\mathbf{u}}\times\vec{\mathbf{v}}$ is perpendicular to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, and is $\vec{\mathbf{0}}$ exactly when $\vec{\mathbf{u}} \parallel \vec{\mathbf{v}}$. Two vectors are perpendicular when their dot product is 0.

2. Three ways to define a plane

A plane in $\mathbb{R}^3$ is uniquely determined by any of the following:

- three **non-collinear points**;
- one **point** and a **nonzero normal vector**;
- a **unit normal** oriented from the origin toward the plane, together with the **distance** p from the origin.

2.1 Three points

Vector (parametric) form

Let A, B, C be non-collinear, $\vec{\mathbf{u}} = \overrightarrow{AB}$ and $\vec{\mathbf{v}} = \overrightarrow{AC}$. Every point P of the plane satisfies

$$\overrightarrow{OP} = \overrightarrow{OA} + s\,\vec{\mathbf{u}} + t\,\vec{\mathbf{v}}, \qquad s, t \in \mathbb{R},$$

that is, $(x, y, z) = A + s(B - A) + t(C - A)$.

Be careful

The points must be **non-collinear**. If $\vec{\mathbf{u}} \parallel \vec{\mathbf{v}}$, the combinations $s\vec{\mathbf{u}} + t\vec{\mathbf{v}}$ only trace a line, and infinitely many planes contain that line.

2.2 A point and a normal

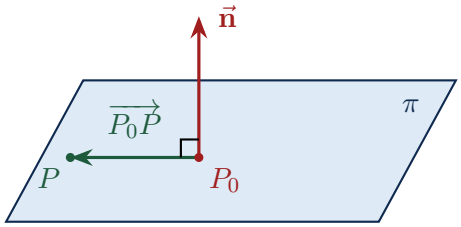
Point–normal form and Cartesian equation

The plane through $P_0(x_0, y_0, z_0)$ with normal $\vec{\mathbf{n}} = (A, B, C) \neq \vec{\mathbf{0}}$ is the set of points P such that

$$\vec{\mathbf{n}} \cdot \overrightarrow{P_0P} = 0 \iff A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Expanding gives the **Cartesian equation**

$$Ax + By + Cz + D = 0, \qquad D = -(Ax_0 + By_0 + Cz_0).$$



Key idea

From $Ax + By + Cz + D = 0$ you can **read the normal instantly**: $\vec{\mathbf{n}} = (A, B, C)$. The normal fixes the *orientation* of the plane; the constant D (equivalently, one point) fixes its *position*. Any nonzero multiple of $\vec{\mathbf{n}}$ is also a normal.

Method

From three points to a Cartesian equation:

1. Compute $\vec{\mathbf{u}} = B - A$ and $\vec{\mathbf{v}} = C - A$.
2. Compute $\vec{\mathbf{n}} = \vec{\mathbf{u}} \times \vec{\mathbf{v}}$ (simplify by a scalar if possible).
3. Write $\vec{\mathbf{n}} \cdot \overrightarrow{AP} = 0$ and expand.
4. Check by substituting B and C.

Example

$A(1, 0, 0)$, $B(0, 1, 0)$, $C(0, 0, 1)$: $\vec{\mathbf{u}} = (-1, 1, 0)$, $\vec{\mathbf{v}} = (-1, 0, 1)$, so $\vec{\mathbf{n}} = \vec{\mathbf{u}} \times \vec{\mathbf{v}} = (1, 1, 1)$. With $P_0 = A$:

$$1(x - 1) + 1(y - 0) + 1(z - 0) = 0 \iff x + y + z - 1 = 0.$$

Check: B gives $0 + 1 + 0 - 1 = 0$ and C gives $0 + 0 + 1 - 1 = 0$.

2.3 A unit normal and a distance (Hesse form)

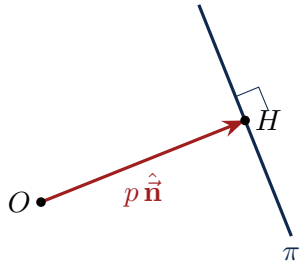
Normal (Hesse) form

Let $\hat{\vec{\mathbf{n}}}$ be a unit normal pointing *from the origin toward the plane* and $p = d(O, \pi) \geq 0$. Then

$$\hat{\vec{\mathbf{n}}} \cdot \overrightarrow{OP} = p \quad \text{for every } P \in \pi.$$

From $Ax + By + Cz + D = 0$, divide by $\|\vec{\mathbf{n}}\|$:

$$\hat{\vec{\mathbf{n}}} = \frac{(A, B, C)}{\|\vec{\mathbf{n}}\|}, \qquad p = \frac{|D|}{\sqrt{A^2 + B^2 + C^2}}.$$



Edge-on view: $H = p\,\hat{\vec{\mathbf{n}}}$ is the point of π closest to O.

Be careful

Choose the sign of $\hat{\vec{\mathbf{n}}}$ so that the right-hand side is ≥ 0: if $D > 0$, flip the signs of both $\vec{\mathbf{n}}$ and D before dividing. If $p = 0$, the plane passes through O and either orientation works.

Example

$\pi : 2x - 2y + z - 6 = 0$. Then $\vec{\mathbf{n}} = (2, -2, 1)$, $\|\vec{\mathbf{n}}\| = 3$ and

$$\frac{2x - 2y + z}{3} = 2, \qquad \hat{\vec{\mathbf{n}}} = \left(\tfrac{2}{3}, -\tfrac{2}{3}, \tfrac{1}{3}\right), \qquad p = \frac{|-6|}{3} = 2.$$

Here $D < 0$, so no sign flip is needed.

2.4 Moving between the descriptions

Roadmap

Given	Procedure
A, B, C non-collinear	$\vec{\mathbf{u}} = B - A$, $\vec{\mathbf{v}} = C - A$; $\vec{\mathbf{n}} = \vec{\mathbf{u}} \times \vec{\mathbf{v}}$; use A and $\vec{\mathbf{n}}$.
Point P_0, normal $\vec{\mathbf{n}}$	Write $\vec{\mathbf{n}} \cdot \overrightarrow{P_0P} = 0$. For a parametric form, take two independent solutions of $\vec{\mathbf{n}} \cdot \vec{\mathbf{u}} = 0$.
$Ax+By+Cz+D = 0$	Read $\vec{\mathbf{n}} = (A, B, C)$; find one point by solving; fix the sign and divide by $\|\vec{\mathbf{n}}\|$ for the Hesse form.
Unit normal $\hat{\vec{\mathbf{n}}}$, p	Expand $\hat{\vec{\mathbf{n}}} \cdot (x, y, z) = p$.

Example — back to a parametric form

$\pi : x + y + z - 3 = 0$, $\vec{\mathbf{n}} = (1, 1, 1)$, $P_0 = (1, 1, 1)$. Any (a, b, c) with $a + b + c = 0$ lies in the plane's direction; take $\vec{\mathbf{u}} = (1, -1, 0)$ and $\vec{\mathbf{v}} = (1, 0, -1)$:

$$(x, y, z) = (1, 1, 1) + s(1, -1, 0) + t(1, 0, -1).$$

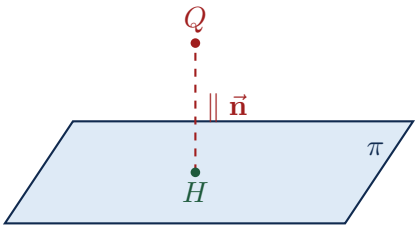
Example — round trip

Through $A(2, 0, 0)$, $B(0, 3, 0)$, $C(0, 0, 6)$: $\overrightarrow{AB} \times \overrightarrow{AC} = (18, 12, 6)$, so $\vec{\mathbf{n}} = (3, 2, 1)$ and $\pi : 3x + 2y + z = 6$. Distance from O: $6/\sqrt{14}$, with unit normal $(3, 2, 1)/\sqrt{14}$. Membership: $D(1, 1, 1)$ gives $3 + 2 + 1 = 6$, so $D \in \pi$.

3. Orthogonal projection of a point on a plane

Key idea

The projection H of Q onto π is reached by moving from Q **parallel to the normal** until we hit the plane. It is the closest point of π to Q: for any $R \in \pi$, $QR^2 = QH^2 + HR^2 \geq QH^2$. If $Q \in \pi$, then $H = Q$.



Projection formula

Let π pass through P_0 with normal $\vec{\mathbf{n}}$, and set $\vec{\mathbf{w}} = Q - P_0$. Then

$$H = Q - t\,\vec{\mathbf{n}}, \qquad t = \frac{\vec{\mathbf{n}} \cdot \vec{\mathbf{w}}}{\vec{\mathbf{n}} \cdot \vec{\mathbf{n}}}.$$

Why. Write $H = Q - t\vec{\mathbf{n}}$. Since $H \in \pi$, $\vec{\mathbf{n}} \cdot \overrightarrow{P_0H} = \vec{\mathbf{n}} \cdot (\vec{\mathbf{w}} - t\vec{\mathbf{n}}) = 0$, so $\vec{\mathbf{n}} \cdot \vec{\mathbf{w}} - t\,\|\vec{\mathbf{n}}\|^2 = 0$.

Be careful

Compare with the projection onto a *line*: there we move **along the direction vector, starting from** P_0; here we move **along the normal, starting from** Q.

Reminder

For $\pi : Ax + By + Cz + D = 0$, the number $\vec{\mathbf{n}} \cdot \vec{\mathbf{w}}$ is simply the left-hand side of the equation evaluated at Q. Its sign tells which side of the plane Q is on.

Example

Project $Q(4,5,6)$ onto $\pi : x+y+z-3=0$ ($P_0=(1,1,1)$, $\vec{\mathbf{n}}=(1,1,1)$):

$$\vec{\mathbf{w}}=(3,4,5), \quad t=\frac{12}{3}=4, \quad H=(4,5,6)-4(1,1,1)=(0,1,2).$$

Check: $0+1+2-3=0$ and $\overrightarrow{HQ}=(4,4,4) \parallel \vec{\mathbf{n}}$.

4. Distance from a point to a plane

Key idea

The distance is $\|H-Q\| = \|-t\vec{\mathbf{n}}\| = |t|\,\|\vec{\mathbf{n}}\|$. Substituting t gives a formula that **never requires computing** H.

Distance formula

For $\pi : Ax+By+Cz+D=0$ and $Q(x_0,y_0,z_0)$:

$$d(Q,\pi)=\frac{|\vec{\mathbf{n}}\cdot\vec{\mathbf{w}}|}{\|\vec{\mathbf{n}}\|}=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}.$$

Special case $Q=O$: $d(O,\pi)=|D|/\sqrt{A^2+B^2+C^2}=p$, the Hesse distance.

Example

$d\big(Q(4,5,6),\ x+y+z-3=0\big)=\dfrac{|4+5+6-3|}{\sqrt{3}}=\dfrac{12}{\sqrt{3}}=4\sqrt{3}$. Cross-check with the projection: $\|Q-H\|=\|(4,4,4)\|=\sqrt{48}=4\sqrt{3}$. Same answer, two routes.

Example

A plane has normal $(0,3,4)$ and passes through $(1,0,0)$, so $\pi : 3y+4z=0$. Then

$$d\big((1,5,5),\pi\big)=\frac{|15+20|}{5}=7.$$

5. Intersection of a line and a plane

Let $L : (x,y,z)=P_1+t\,\vec{\mathbf{d}}$ and $\pi : Ax+By+Cz+D=0$ with normal $\vec{\mathbf{n}}$.

Three possible relationships

Case	Test	Intersection
Transversal	$\vec{\mathbf{n}} \cdot \vec{\mathbf{d}} \neq 0$	exactly one point
Parallel, disjoint	$\vec{\mathbf{n}} \cdot \vec{\mathbf{d}} = 0$ and $P_1 \notin \pi$	empty
Contained	$\vec{\mathbf{n}} \cdot \vec{\mathbf{d}} = 0$ and $P_1 \in \pi$	every point of L

Key idea

$\vec{\mathbf{n}} \cdot \vec{\mathbf{d}}$ measures how much the line points *out of* the plane. It vanishes exactly when $\vec{\mathbf{d}}$ is a direction of the plane; the starting point then decides between "disjoint" and "contained".

Method

Substitute and solve:

1. Substitute the parametric coordinates of L into the equation of π.
2. Solve the resulting single linear equation in t.
3. Unique t: one point (plug it back into L). Identity $0 = 0$: $L \subset \pi$. False statement (e.g. $-3 = 0$): $L \parallel \pi$, no intersection.

Examples with $\pi : x + y + z - 3 = 0$

- $L = (1, 0, -1) + t(2, 1, 3)$: $(1 + 2t) + t + (-1 + 3t) - 3 = 6t - 3 = 0$, so $t = \frac{1}{2}$ and the point is $\left(2, \frac{1}{2}, \frac{1}{2}\right)$.
- $L = (0, 0, 0) + t(1, -1, 0)$: $t - t + 0 - 3 = 0$ gives $-3 = 0$. Parallel and disjoint.
- $L = (1, 1, 1) + t(1, -1, 0)$: $(1 + t) + (1 - t) + 1 - 3 = 0$ gives $0 = 0$. The line lies in π.

Be careful

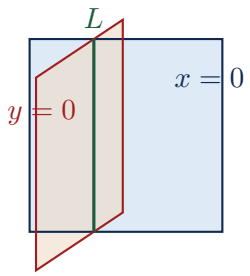
In a sketch, a line may seem to cross a plane even when they are disjoint in space. Classify with $\vec{\mathbf{n}} \cdot \vec{\mathbf{d}}$ and the starting point, not with the picture.

6. Intersection of two planes

Let π_1, π_2 have normals $\vec{\mathbf{n}}_1, \vec{\mathbf{n}}_2$.

Key idea

Two planes are either **equal**, **parallel and distinct** (no intersection), or **transversal** (they meet in a line). Unlike two lines in $\mathbb{R}^3$, two planes can never be skew: if their normals are not parallel, they *must* cross.



The planes $x = 0$ and $y = 0$ meet along the z-axis, $L : (x, y, z) = (0, 0, t)$.

Method

1. Compute $\vec{\mathbf{n}}_1 \times \vec{\mathbf{n}}_2$.
2. If $\vec{\mathbf{n}}_1 \times \vec{\mathbf{n}}_2 = \vec{\mathbf{0}}$ (parallel normals): take a point of π_1 and test it in π_2. Satisfied: **equal planes**. Not satisfied: **parallel and distinct**.
3. If $\vec{\mathbf{n}}_1 \times \vec{\mathbf{n}}_2 \neq \vec{\mathbf{0}}$: the intersection is a line with direction $\vec{\mathbf{d}} = \vec{\mathbf{n}}_1 \times \vec{\mathbf{n}}_2$. To get a point, fix one coordinate (e.g. $z = 0$) and solve the 2×2 system by substitution. If that choice gives no solution, fix another coordinate.

Be careful

Parallel normals alone do not distinguish equal planes from disjoint ones. $x + 2y - z - 1 = 0$ and $2x+4y-2z-10 = 0$ are **parallel and distinct**, whereas $x+2y-z-1 = 0$ and $2x+4y-2z-2 = 0$ are **equal**: the whole equation, constant included, is a multiple.

Example

$\pi_1 : x + y + z - 3 = 0$, $\pi_2 : x - y + 2z - 2 = 0$.

$$\vec{\mathbf{d}} = (1, 1, 1) \times (1, -1, 2) = (3, -1, -2) \neq \vec{\mathbf{0}}.$$

Check: $\vec{\mathbf{n}}_1 \cdot \vec{\mathbf{d}} = 0$ and $\vec{\mathbf{n}}_2 \cdot \vec{\mathbf{d}} = 0$. With $z = 0$: $x + y = 3$, $x - y = 2$, so $x = \frac{5}{2}$, $y = \frac{1}{2}$. Hence

$$L : (x, y, z) = \left(\tfrac{5}{2}, \tfrac{1}{2}, 0\right) + t(3, -1, -2), \qquad t \in \mathbb{R}.$$

Substituting L into both equations gives identities in t: the *whole* line lies in both planes.

7. Summary and common pitfalls

Formulas to remember

Normal from three points	$\vec{\mathbf{n}} = \overrightarrow{AB} \times \overrightarrow{AC}$
Cartesian equation	$Ax + By + Cz + D = 0, \quad \vec{\mathbf{n}} = (A, B, C)$
Distance from the origin	$p = \dfrac{\lvert D \rvert}{\sqrt{A^2 + B^2 + C^2}}$
Projection of Q	$H = Q - \dfrac{\vec{\mathbf{n}} \cdot \vec{\mathbf{w}}}{\vec{\mathbf{n}} \cdot \vec{\mathbf{n}}}\,\vec{\mathbf{n}}, \quad \vec{\mathbf{w}} = Q - P_0$
Distance from Q	$d(Q, \pi) = \dfrac{\lvert Ax_0 + By_0 + Cz_0 + D \rvert}{\sqrt{A^2 + B^2 + C^2}}$
Line $\cap$ plane	transversal $\iff \vec{\mathbf{n}} \cdot \vec{\mathbf{d}} \neq 0$
Plane $\cap$ plane	direction of the line: $\vec{\mathbf{d}} = \vec{\mathbf{n}}_1 \times \vec{\mathbf{n}}_2$

Be careful

- Using collinear points: the cross product is $\vec{\mathbf{0}}$ and gives no normal.
- Forgetting the absolute value in the distance formula.
- Projecting from P_0 instead of from Q: $H = Q - t\vec{\mathbf{n}}$, not $P_0 - t\vec{\mathbf{n}}$.
- Concluding "equal planes" from parallel normals alone: always test a point.
- Checking only one point of an intersection line: verify that the identity holds for *every* t.

Method

Systematic checks: a point is on a plane if it satisfies the equation; a direction is in a plane if its dot product with $\vec{\mathbf{n}}$ is 0; a projection satisfies $H \in \pi$ and $\overrightarrow{HQ} \parallel \vec{\mathbf{n}}$; a distance can be computed both by the formula and by $\|Q - H\|$.