Chapter 4 - Exercises

Planes in Space – Equations, Projection, Distance, and Intersections

Collège Sainte-Anne | Fall 2026

Quick Concept Check

True or false? Justify each answer. For a false statement, give a correction or a counterexample.

Exercise 1. A plane has exactly one normal vector.

Exercise 2. Any three distinct points determine a unique plane.

Exercise 3. Two planes with parallel normal vectors must be identical.

Exercise 4. A line with direction $\vec{\mathbf{d}}$ intersects a plane with normal $\vec{\mathbf{n}}$ in exactly one point if $\vec{\mathbf{n}} \cdot \vec{\mathbf{d}} \neq 0$.

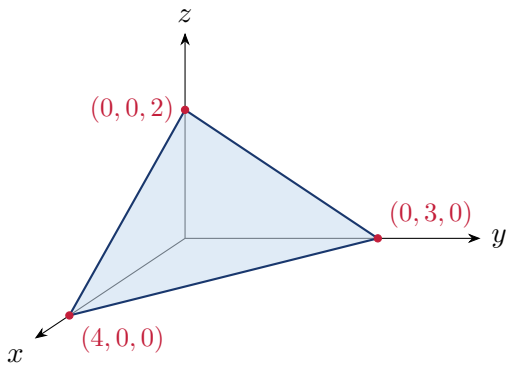
Exercise 5. Two planes in $\mathbb{R}^3$ can be skew, as two lines can be.

Exercise 6. For $Ax + By + Cz + D = 0$, with $(A, B, C) \neq (0, 0, 0)$, the value $Ax_0 + By_0 + Cz_0 + D$ is the distance from (x_0, y_0, z_0) to the plane.

Visual and Geometric Reasoning

A shaded patch represents only part of an infinite plane. Perspective does not preserve angles or lengths.

Exercise 7. The plane below meets the axes exactly at the three marked points.



(a) Write a Cartesian equation of the plane from its intercepts.

(b) Read off a normal vector from your equation.

(c) Without projecting, decide whether $(2, 1, 1)$ lies on the plane, on the origin's side, or on the opposite side from the origin. Justify using your equation from (a), written with all terms on the left, and compare the signs at the two points.

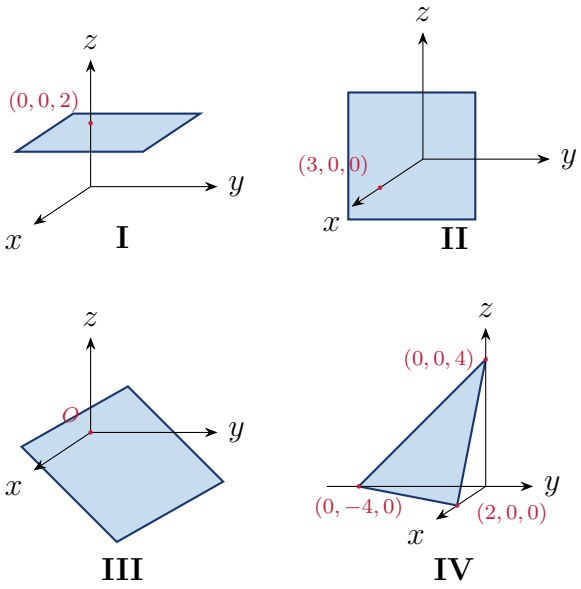
Exercise 8. A plane π passes through P_0 with normal $\vec{\mathbf{n}}$.

(a) Explain why $\vec{\mathbf{n}} \cdot \overrightarrow{P_0P} = 0$ for every $P \in \pi$.

(b) Explain the difference between a point on π, a direction vector of π, and a normal vector to π.

(c) A student says: "The normal vector gives a direction along the plane." Correct this statement.

Exercise 9. Match each sketch to an equation below. The marked coordinates are exact. For each match, cite a geometric feature that supports your choice.



(a) $z = 2$ (b) $x = 3$ (c) $x + y + z = 0$ (d) $2x - y + z = 4$.

Building Equations

Start with a normal and a point, then combine geometric data.

Exercise 10. A plane passes through $P_0(2, -1, 3)$ and has normal $\vec{\mathbf{n}} = (1, 2, -2)$.

(a) Write its point-normal and Cartesian equations.

(b) Decide which of $A(0, 0, 3)$ and $B(2, 0, 3)$ lies on the plane.

(c) Is $\vec{\mathbf{u}} = (2, -1, 0)$ a direction vector of this plane? Explain why substituting the coordinates of a vector into the plane's equation is not the correct test in general.

Exercise 11. Let $A(1,0,1)$, $B(2,1,0)$, and $C(0,2,3)$.

(a) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and show that the three points are noncollinear.

(b) Find a normal vector and a Cartesian equation of their plane.

(c) Write a vector equation of the same plane, using two parameters.

(d) Verify that all three points satisfy your Cartesian equation. Does the origin lie on the plane?

Exercise 12. A plane passes through $P_0(-1,2,0)$ with normal $\vec{\mathbf{n}} = (2,0,-3)$.

(a) Write its Cartesian equation. Explain geometrically why y does not appear.

(b) Find two additional points on the plane.

(c) Find two nonparallel direction vectors and give a vector equation.

Exercise 13. Consider $6x - 3y + 2z - 12 = 0$.

(a) Write its Hesse normal form $\widehat{\vec{\mathbf{n}}} \cdot (x,y,z) = p$, choosing the unit normal so that $p \geq 0$. Interpret p geometrically.

(b) Find a point and two nonparallel direction vectors; write a vector equation of the plane.

(c) Explain why one parameter would describe only a line with fixed direction, rather than this entire plane.

Exercise 14. Let $L : (x,y,z) = (1,0,2) + t(1,-1,3)$ and $Q(0,4,-1)$.

(a) Show that $Q \notin L$. Explain why L and Q determine a unique plane.

(b) Find a Cartesian equation of that plane.

(c) Verify that your equation holds for every point of L and for Q.

Projection and Distance

Distinguish the signed expression in an equation from a geometric distance.

Exercise 15. Let $Q(3,-1,4)$ and $\pi : x - 2y + 2z - 9 = 0$.

(a) Find the orthogonal projection H of Q onto π.

(b) Check that $H \in \pi$ and that $\overrightarrow{HQ}$ is parallel to a normal vector.

(c) Compute $d(Q,\pi)$ using both QH and the distance formula.

Exercise 16. A plane has unit normal $\widehat{\vec{\mathbf{n}}} = (\frac{2}{3}, -\frac{2}{3}, \frac{1}{3})$ and contains $(3, 3, 3)$.

(a) Find its Hesse normal form and its distance from the origin.

(b) Write a Cartesian equation with integer coefficients.

(c) Find the distance from $(6, 0, 0)$ to this plane.

Intersections

Predict using normals and direction vectors, then confirm by substitution.

Exercise 17. For $\pi : 3x - y + z - 10 = 0$, classify each line as intersecting π at one point, parallel and disjoint, or contained in π.

(a) $L_1 : (x, y, z) = (2, -1, 3) + t(1, 2, -1)$.

(b) $L_2 : (x, y, z) = t(1, 2, -1)$.

(c) $L_3 : (x, y, z) = (2, -1, 3) + t(0, 0, 1)$.

Show what substitution gives in each case. Explain why $\vec{\mathbf{n}} \cdot \vec{\mathbf{d}} = 0$ alone does not distinguish (a) from (b).

Exercise 18. Let $L : (x, y, z) = (0, 1, -2) + t(2, -1, 1)$ and $\pi : x + y + z = 5$.

(a) Show that there is exactly one intersection point.

(b) Find that point and check that it belongs to both L and π.