Chapter 4 - Solutions

Planes in Space – Equations, Projection, Distance, and Intersections

Collège Sainte-Anne | Fall 2026

Quick Concept Check

Exercise 1.

A plane has exactly one normal vector.

False. If $\vec{\mathbf{n}}$ is normal to π, then $\vec{\mathbf{n}} \cdot \vec{\mathbf{u}} = 0$ for every direction $\vec{\mathbf{u}}$ of π; hence $(k\vec{\mathbf{n}}) \cdot \vec{\mathbf{u}} = k(\vec{\mathbf{n}} \cdot \vec{\mathbf{u}}) = 0$ for every scalar k. So *every nonzero multiple* $k\vec{\mathbf{n}}$ is also a normal.

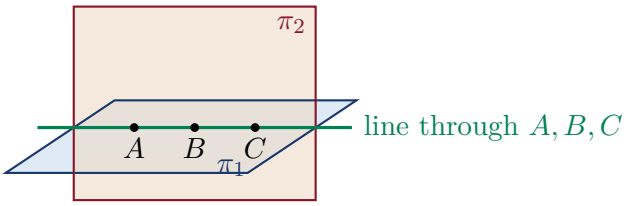
Correction

A plane has infinitely many normal vectors, all parallel to each other. It has a unique normal *direction* (and exactly two *unit* normals, $\pm\hat{\vec{\mathbf{n}}}$). Example: $x + y + z = 1$ has normals $(1, 1, 1)$, $(2, 2, 2)$, $(-1, -1, -1)$, ...

Exercise 2.

Any three distinct points determine a unique plane.

False. The points must be *non-collinear*. If they lie on one line, every plane containing that line contains all three.



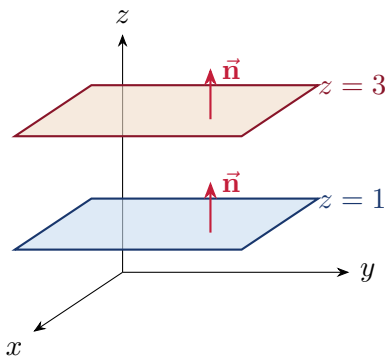
Counterexample

$A(0, 1, 0)$, $B(0, 2, 0)$, $C(0, 3, 0)$ are distinct but collinear (all on the y-axis). Both $\pi_1 : z = 0$ and $\pi_2 : x = 0$ contain them, and so does every plane $ax + cz = 0$. Algebraically: $\overrightarrow{AB} \times \overrightarrow{AC} = (0, 1, 0) \times (0, 2, 0) = \vec{\mathbf{0}}$, so no normal can be produced. Correct statement: *three non-collinear points determine a unique plane.*

Exercise 3.

Two planes with parallel normal vectors must be identical.

False. Parallel normals only mean the planes have the same *orientation*; their *positions* may differ.



Counterexample

$\pi_1 : z = 1$ and $\pi_2 : z = 3$ both have normal $(0, 0, 1)$, yet $(0, 0, 1) \in \pi_1$ and $(0, 0, 1) \notin \pi_2$. They are parallel and distinct (no common point). Parallel normals give two possibilities, *equal* or *parallel and distinct*; to decide, test one point of π_1 in the equation of π_2.

Exercise 4.

A line with direction $\vec{\mathbf{d}}$ intersects a plane with normal $\vec{\mathbf{n}}$ in exactly one point if $\vec{\mathbf{n}} \cdot \vec{\mathbf{d}} \neq 0$.

True. Let $L : P_1 + t\vec{\mathbf{d}}$ and $\pi : \vec{\mathbf{n}} \cdot (x, y, z) + D = 0$. Substituting the line into the plane gives

$$\vec{\mathbf{n}} \cdot (P_1 + t\vec{\mathbf{d}}) + D = 0 \iff (\vec{\mathbf{n}} \cdot \vec{\mathbf{d}})\, t = -(\vec{\mathbf{n}} \cdot P_1 + D).$$

This is a linear equation in t whose coefficient $\vec{\mathbf{n}} \cdot \vec{\mathbf{d}}$ is nonzero, so it has exactly one solution

$$t^* = -\frac{\vec{\mathbf{n}} \cdot P_1 + D}{\vec{\mathbf{n}} \cdot \vec{\mathbf{d}}},$$

and L meets π at the single point $P_1 + t^*\vec{\mathbf{d}}$.

Answer

True. Geometrically, $\vec{\mathbf{n}} \cdot \vec{\mathbf{d}} \neq 0$ means $\vec{\mathbf{d}}$ is not a direction of the plane, so the line "pierces" π. (The converse also holds: when $\vec{\mathbf{n}} \cdot \vec{\mathbf{d}} = 0$ there are either 0 or infinitely many intersection points.)

Exercise 5.

Two planes in $\mathbb{R}^3$ can be skew, as two lines can be.

False. Take π_1, π_2 with normals $\vec{\mathbf{n}}_1, \vec{\mathbf{n}}_2$.

- If $\vec{\mathbf{n}}_1 \parallel \vec{\mathbf{n}}_2$: the planes are equal or parallel and distinct.
- If $\vec{\mathbf{n}}_1 \nparallel \vec{\mathbf{n}}_2$: then $\vec{\mathbf{d}} = \vec{\mathbf{n}}_1 \times \vec{\mathbf{n}}_2 \neq \vec{\mathbf{0}}$ is a direction of both planes, and they always share a point (fixing a suitable coordinate leads to a 2×2 system with exactly one solution). So they meet in a whole line.

Answer

False. "Skew" means *not parallel and not intersecting*; for planes this cannot happen: non-parallel planes always meet in a line. Lines can be skew because a line has only one direction, which leaves "room" to miss another line; a plane has two independent directions and cannot avoid a non-parallel plane.

Exercise 6.

For $Ax + By + Cz + D = 0$, with $(A, B, C) \neq (0, 0, 0)$, the value $Ax_0 + By_0 + Cz_0 + D$ is the distance from (x_0, y_0, z_0) to the plane.

False, for two separate reasons.

(a) **It can be negative**, while a distance is always ≥ 0. Its sign only tells which side of the plane the point is on.

(b) **It depends on the scaling of the equation.** Multiplying the equation by k multiplies this value by k, but the plane (and the distance) does not change. We must divide by $\|\vec{\mathbf{n}}\| = \sqrt{A^2 + B^2 + C^2}$.

Counterexample and correction

For $\pi : 2x = 0$ (the plane $x = 0$) and the point $(-1, 0, 0)$, the value is $2(-1) = -2$, whereas the distance is 1. Correct formula:

$$d\Big((x_0, y_0, z_0), \pi\Big) = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

The value equals the distance exactly in two situations: when the point lies *on* the plane (both are 0, whatever the scaling), or when $\|\vec{\mathbf{n}}\| = 1$ and the point lies on the side toward which $\vec{\mathbf{n}}$ points.

Visual and Geometric Reasoning

Exercise 7.

The plane meets the axes at $(4,0,0)$, $(0,3,0)$, $(0,0,2)$. (a) Cartesian equation from the intercepts. (b) A normal vector. (c) Is $(2,1,1)$ on the plane, on the origin's side, or on the opposite side?

(a) A plane with intercepts a, b, c on the three axes (all nonzero) has the *intercept form* $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. Here

$$\frac{x}{4}+\frac{y}{3}+\frac{z}{2}=1 \quad \overset{\times 12}{\Longleftrightarrow} \quad 3x+4y+6z=12 \iff 3x+4y+6z-12=0.$$

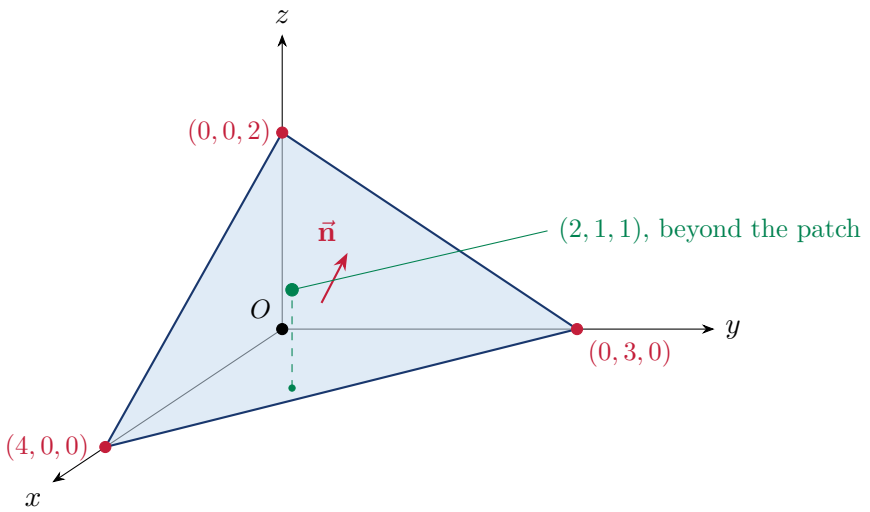
Check the three points: $3(4)=12$, $4(3)=12$, $6(2)=12$ ✓.

(b) Reading the coefficients: $\vec{\mathbf{n}}=(3,4,6)$ (or any nonzero multiple).

(c) Let $f(x,y,z)=3x+4y+6z-12$. Points with $f=0$ lie on the plane; points with the same sign of f lie on the same side.

$$f(0,0,0)=-12<0, \qquad f(2,1,1)=6+4+6-12=4>0.$$

The signs are opposite, so $(2,1,1)$ is **on the opposite side from the origin**.



$f=-12$ at O and $f=+4$ at $(2,1,1)$: the plane separates them.

Answer

(a) $3x+4y+6z-12=0$ (b) $\vec{\mathbf{n}}=(3,4,6)$ (c) opposite side from O, since $f(O)=-12<0<4=f(2,1,1)$. Moreover $d\Big((2,1,1),\pi\Big)=4/\sqrt{61}$, so the point is close to the plane.

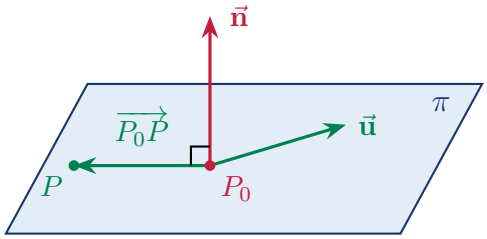
Why the sign test works

$f(Q)=\vec{\mathbf{n}}\cdot(Q-P_0)$ for any $P_0\in\pi$. It is positive when $\overrightarrow{P_0Q}$ points to the side $\vec{\mathbf{n}}$ points to, negative on the other side. Points on the same side share a sign.

Exercise 8.

A plane π passes through P_0 with normal $\vec{\mathbf{n}}$. (a) Why is $\vec{\mathbf{n}} \cdot \overrightarrow{P_0P} = 0$ for every $P \in \pi$? (b) Point vs. direction vector vs. normal vector. (c) Correct: "The normal vector gives a direction along the plane."

(a) If $P \in \pi$, both endpoints of $\overrightarrow{P_0P}$ lie in π, so $\overrightarrow{P_0P}$ is a vector *lying in* the plane. By definition the normal is perpendicular to every vector lying in the plane, and perpendicular vectors have zero dot product. Conversely, if $\vec{\mathbf{n}} \cdot \overrightarrow{P_0P} = 0$ then P is in π: the equation characterizes the plane exactly.



(b)

- A **point** of π is a *location*: its coordinates satisfy the Cartesian equation. It fixes *where* the plane is.
- A **direction vector** of π is a *displacement* you can make while staying in the plane (e.g. $\overrightarrow{P_0P}$ or $\vec{\mathbf{u}}$ above). It satisfies $\vec{\mathbf{n}} \cdot \vec{\mathbf{u}} = 0$. A plane has two independent ones.
- A **normal vector** is a displacement *perpendicular* to the plane. It fixes the *orientation* but not the position: every plane parallel to π has the same normals.

(c) The normal is perpendicular to the plane, not along it. Moving from a point of π in the direction of $\vec{\mathbf{n}}$ leaves the plane immediately (except for the zero displacement).

Correct statement

"The normal vector gives the direction *perpendicular* to the plane; the directions *along* the plane are exactly the vectors $\vec{\mathbf{u}}$ with $\vec{\mathbf{n}} \cdot \vec{\mathbf{u}} = 0$."

Exercise 9.

Match sketches I–IV to (a) $z = 2$, (b) $x = 3$, (c) $x + y + z = 0$, (d) $2x - y + z = 4$, citing a geometric feature.

Sketch	Equation	Supporting feature
I	(a) $z = 2$	Horizontal patch through $(0, 0, 2)$: every point has $z = 2$; parallel to the xy-plane, normal $(0, 0, 1)$.
II	(b) $x = 3$	Patch perpendicular to the x-axis through $(3, 0, 0)$: every point has $x = 3$; parallel to the yz-plane, normal $(1, 0, 0)$.
III	(c) $x + y + z = 0$	Passes through the origin O, which forces $D = 0$; only (c) has $D = 0$. Each drawn corner satisfies it, e.g. $(1.5, -1.5, 0)$: $1.5 - 1.5 + 0 = 0$.
IV	(d) $2x - y + z = 4$	Intercepts $(2, 0, 0)$, $(0, -4, 0)$, $(0, 0, 4)$: $2(2) = 4$, $-(-4) = 4$, $4 = 4$ ✓. The negative y-intercept matches the $-y$ term.

Elimination strategy

A missing variable means the plane is parallel to that axis: $z = 2$ lacks x and y (parallel to both), $x = 3$ lacks y and z. A zero constant means the plane contains O. Intercepts are read by setting two coordinates to 0.

Building Equations

Exercise 10.

Plane through $P_0(2, -1, 3)$ with normal $\vec{\mathbf{n}} = (1, 2, -2)$. (a) Point-normal and Cartesian equations. (b) Which of $A(0, 0, 3)$, $B(2, 0, 3)$ lies on it? (c) Is $\vec{\mathbf{u}} = (2, -1, 0)$ a direction vector? Why is substituting a vector into the equation not the right test?

(a) Point-normal form $\vec{\mathbf{n}} \cdot \overrightarrow{P_0P} = 0$:

$$1(x - 2) + 2(y + 1) - 2(z - 3) = 0.$$

Expanding: $x - 2 + 2y + 2 - 2z + 6 = 0$, so

$$\boxed{x + 2y - 2z + 6 = 0.}$$

Check with P_0: $2 - 2 - 6 + 6 = 0$ ✓.

(b) $A(0, 0, 3)$: $0 + 0 - 6 + 6 = 0$, so $A \in \pi$. $B(2, 0, 3)$: $2 + 0 - 6 + 6 = 2 \neq 0$, so $B \notin \pi$.

(c) A vector $\vec{\mathbf{u}}$ is a direction of π iff $\vec{\mathbf{u}} \perp \vec{\mathbf{n}}$:

$$\vec{\mathbf{n}} \cdot \vec{\mathbf{u}} = (1)(2) + (2)(-1) + (-2)(0) = 0.$$

So **yes**, $\vec{\mathbf{u}}$ is a direction vector. Indeed $\overrightarrow{AP_0} = (2, -1, 0) = \vec{\mathbf{u}}$ joins two points of the plane.

Substituting $\vec{\mathbf{u}}$ into the equation gives $2 - 2 - 0 + 6 = 6 \neq 0$, which would wrongly suggest "no". The equation $x + 2y - 2z + 6 = 0$ tests *points* (locations); it contains the constant $D = 6$ that encodes the plane's position. A direction vector is a displacement, independent of position, and must satisfy the *homogeneous* condition $\vec{\mathbf{n}} \cdot \vec{\mathbf{u}} = 0$, i.e. the equation *without* D.

Answer

(a) $(x-2) + 2(y+1) - 2(z-3) = 0$, i.e. $x + 2y - 2z + 6 = 0$. (b) $A \in \pi$, $B \notin \pi$. (c) Yes, since $\vec{\mathbf{n}} \cdot \vec{\mathbf{u}} = 0$. Substitution into the full equation is valid only when $D = 0$ (plane through O), where points and directions coincide.

Exercise 11.

$A(1,0,1)$, $B(2,1,0)$, $C(0,2,3)$. (a) $\overrightarrow{AB}$, $\overrightarrow{AC}$; non-collinearity. (b) Normal and Cartesian equation. (c) Vector equation. (d) Check the three points; is O on the plane?

(a)

$$\overrightarrow{AB} = B - A = (1, 1, -1), \qquad \overrightarrow{AC} = C - A = (-1, 2, 2).$$

They are not parallel: if $(-1, 2, 2) = k(1, 1, -1)$, the first coordinate gives $k = -1$, but the second gives $k = 2$. Contradiction. Hence A, B, C are non-collinear (equivalently, the cross product below is nonzero).

(b)

$$\vec{\mathbf{n}} = \overrightarrow{AB} \times \overrightarrow{AC} = \Big(1 \cdot 2 - (-1) \cdot 2,\ (-1)(-1) - 1 \cdot 2,\ 1 \cdot 2 - 1 \cdot (-1)\Big) = (4, -1, 3).$$

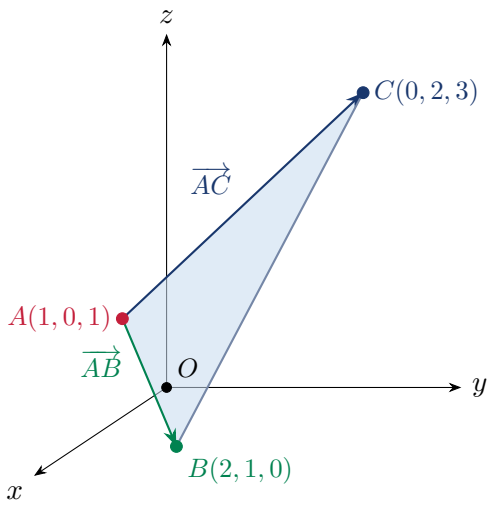
With $P_0 = A$: $4(x-1) - (y-0) + 3(z-1) = 0$, so

$$\boxed{4x - y + 3z - 7 = 0.}$$

Check that $\vec{\mathbf{n}}$ is perpendicular to both: $4 - 1 - 3 = 0$ and $-4 - 2 + 6 = 0$ ✓.

(c) $(x, y, z) = (1, 0, 1) + s(1, 1, -1) + t(-1, 2, 2), \quad s, t \in \mathbb{R}$.

(d) A: $4 - 0 + 3 - 7 = 0$ ✓; B: $8 - 1 + 0 - 7 = 0$ ✓; C: $0 - 2 + 9 - 7 = 0$ ✓. Origin: $0 - 0 + 0 - 7 = -7 \neq 0$, so $O \notin \pi$.



Answer

$\vec{\mathbf{n}} = (4, -1, 3)$, $4x - y + 3z - 7 = 0$, $(x, y, z) = (1, 0, 1) + s(1, 1, -1) + t(-1, 2, 2)$; $O \notin \pi$ and $d(O, \pi) = 7/\sqrt{26}$.

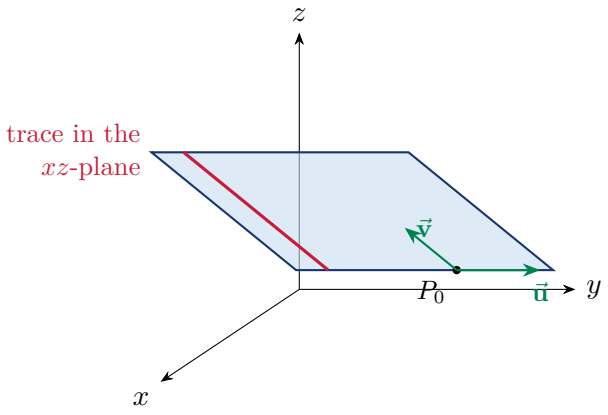
Exercise 12.

Plane through $P_0(-1, 2, 0)$ with normal $\vec{\mathbf{n}} = (2, 0, -3)$. (a) Cartesian equation; why is y absent? (b) Two more points. (c) Two non-parallel direction vectors and a vector equation.

(a) $2(x + 1) + 0(y - 2) - 3(z - 0) = 0$, so

$$\boxed{2x - 3z + 2 = 0.}$$

The y-component of $\vec{\mathbf{n}}$ is 0, so $\vec{\mathbf{n}} \perp (0, 1, 0)$: the y-direction is a direction of the plane. The plane is therefore **parallel to the y-axis**: if (x, y, z) is on it, so is (x, y', z) for every y', and the equation places no constraint on y. It is swept out by sliding the line $2x - 3z + 2 = 0$ of the xz-plane (its trace) parallel to the y-axis. It is tilted, not vertical: its normal has a nonzero z-component.



(b) Choose values and solve $2x - 3z + 2 = 0$ (with y free): $x = -1 \Rightarrow z = 0$: $(-1, 0, 0)$; $x = 2 \Rightarrow z = 2$: $(2, 5, 2)$ (any y works). Checks: $-2 - 0 + 2 = 0$ ✓, $4 - 6 + 2 = 0$ ✓.

(c) We need (a, b, c) with $2a - 3c = 0$. Take $\vec{\mathbf{u}} = (0, 1, 0)$ and $\vec{\mathbf{v}} = (3, 0, 2)$ ($6 - 6 = 0$ ✓). They are not parallel (one has a zero x-component, the other does not). Hence

$$(x, y, z) = (-1, 2, 0) + s(0, 1, 0) + t(3, 0, 2), \qquad s, t \in \mathbb{R}.$$

Answer

$2x - 3z + 2 = 0$; plane parallel to the y-axis. Points e.g. $(-1, 0, 0)$, $(2, 5, 2)$. $(x, y, z) = (-1, 2, 0) + s(0, 1, 0) + t(3, 0, 2)$.

Exercise 13.

$6x - 3y + 2z - 12 = 0$. (a) Hesse normal form with $p \geq 0$; meaning of p. (b) A point, two non-parallel directions, a vector equation. (c) Why would one parameter describe only a line?

(a) $\vec{\mathbf{n}} = (6, -3, 2)$, $\|\vec{\mathbf{n}}\| = \sqrt{36 + 9 + 4} = 7$. Move the constant: $6x - 3y + 2z = 12$, the right-hand side is already positive. Divide by 7:

$$\boxed{\left(\tfrac{6}{7}, -\tfrac{3}{7}, \tfrac{2}{7}\right) \cdot (x, y, z) = \tfrac{12}{7}}, \qquad \hat{\vec{\mathbf{n}}} = \left(\tfrac{6}{7}, -\tfrac{3}{7}, \tfrac{2}{7}\right), \quad p = \tfrac{12}{7}.$$

$p = 12/7$ is the **distance from the origin to the plane**, and $p\,\hat{\vec{\mathbf{n}}} = \left(\frac{72}{49}, -\frac{36}{49}, \frac{24}{49}\right)$ is the point of the plane closest to O (the foot of the perpendicular from O).

(b) Point: setting $y = z = 0$ gives $x = 2$, so $P_0 = (2, 0, 0)$. Directions: solve $6a - 3b + 2c = 0$. $\vec{\mathbf{u}} = (1, 2, 0)$ ($6 - 6 = 0$ ✓) and $\vec{\mathbf{v}} = (0, 2, 3)$ ($-6 + 6 = 0$ ✓), not parallel. So

$$(x, y, z) = (2, 0, 0) + s(1, 2, 0) + t(0, 2, 3), \qquad s, t \in \mathbb{R}.$$

(c) With one parameter, $(2, 0, 0) + s\vec{\mathbf{u}}$ only moves along the single fixed direction $\vec{\mathbf{u}}$: it traces a *line* inside the plane. A plane is two-dimensional: to reach every point we need combinations $s\vec{\mathbf{u}} + t\vec{\mathbf{v}}$ of two independent directions. For instance, $(2, 2, 3) = (2, 0, 0) + (0, 2, 3)$ lies in the plane ($12 - 6 + 6 - 12 = 0$) but is not of the form $(2, 0, 0) + s(1, 2, 0)$.

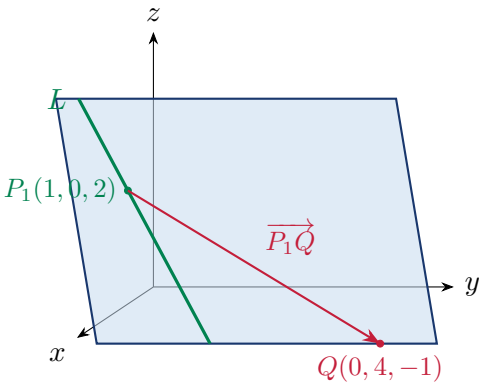
Answer

$\hat{\vec{\mathbf{n}}} \cdot (x, y, z) = 12/7$ with $\hat{\vec{\mathbf{n}}} = \frac{1}{7}(6, -3, 2)$; $d(O, \pi) = 12/7$. $(x, y, z) = (2, 0, 0) + s(1, 2, 0) + t(0, 2, 3)$.

Exercise 14.

$L : (x, y, z) = (1, 0, 2) + t(1, -1, 3)$ and $Q(0, 4, -1)$. (a) $Q \notin L$; why a unique plane? (b) Cartesian equation. (c) Verify for every point of L and for Q.

(a) If $Q \in L$: $1 + t = 0$, $-t = 4$, $2 + 3t = -1$. The first equation gives $t = -1$, the second $t = -4$: contradiction, so $Q \notin L$. Two distinct points of L (e.g. $t = 0$ and $t = 1$) together with Q are three non-collinear points (since Q is off the line), which determine a unique plane. Equivalently, the plane must contain the direction $\vec{\mathbf{d}} = (1, -1, 3)$ and the vector $\overrightarrow{P_1Q}$, and these are not parallel.



(b) $P_1 = (1, 0, 2)$, $\vec{\mathbf{d}} = (1, -1, 3)$, $\overrightarrow{P_1Q} = (-1, 4, -3)$.

$$\vec{\mathbf{d}} \times \overrightarrow{P_1Q} = \big((-1)(-3) - 3 \cdot 4,\ 3(-1) - 1(-3),\ 1 \cdot 4 - (-1)(-1)\big) = (-9, 0, 3) = -3\,(3, 0, -1).$$

Take $\vec{\mathbf{n}} = (3, 0, -1)$. With P_1: $3(x - 1) - (z - 2) = 0$, so

$$\boxed{3x - z - 1 = 0.}$$

(c) Every point of L: $3(1 + t) - (2 + 3t) - 1 = 3 + 3t - 2 - 3t - 1 = 0$ for *all* t ✓. The point Q: $3(0) - (-1) - 1 = 0$ ✓.

Answer

$3x - z - 1 = 0$ (a plane parallel to the y-axis). The identity in t shows the entire line lies in it.

Projection and Distance

Exercise 15.

$Q(3,-1,4)$ and $\pi : x-2y+2z-9=0$. (a) Projection H. (b) Check $H \in \pi$ and $\overrightarrow{HQ} \parallel \vec{\mathbf{n}}$. (c) $d(Q,\pi)$ two ways.

(a) $\vec{\mathbf{n}} = (1,-2,2)$, $\vec{\mathbf{n}} \cdot \vec{\mathbf{n}} = 9$. Take $P_0 = (9,0,0) \in \pi$; then $\vec{\mathbf{w}} = Q - P_0 = (-6,-1,4)$ and

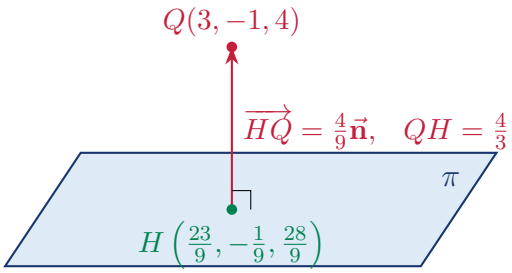
$$\vec{\mathbf{n}} \cdot \vec{\mathbf{w}} = -6 + 2 + 8 = 4$$

(equivalently, the left-hand side at Q: $3+2+8-9=4$). Hence

$$t = \frac{\vec{\mathbf{n}} \cdot \vec{\mathbf{w}}}{\vec{\mathbf{n}} \cdot \vec{\mathbf{n}}} = \frac{4}{9}, \qquad H = Q - t\vec{\mathbf{n}} = (3,-1,4) - \tfrac{4}{9}(1,-2,2) = \left(\tfrac{23}{9}, -\tfrac{1}{9}, \tfrac{28}{9}\right).$$

(b) $H \in \pi$: $\frac{23}{9} - 2\left(-\frac{1}{9}\right) + 2 \cdot \frac{28}{9} - 9 = \frac{23+2+56-81}{9} = 0$ ✓. $\overrightarrow{HQ} = Q - H = \frac{4}{9}(1,-2,2) = \frac{4}{9}\vec{\mathbf{n}}$ ✓.

(c) $QH = \|\overrightarrow{HQ}\| = \frac{4}{9}\|\vec{\mathbf{n}}\| = \frac{4}{9} \cdot 3 = \frac{4}{3}$. Formula: $d(Q,\pi) = \dfrac{|3+2+8-9|}{\sqrt{1+4+4}} = \dfrac{4}{3}$. Same value ✓.



Schematic (not to scale). Q is on the side $\vec{\mathbf{n}}$ points to, because $f(Q) = 4 > 0$.

Answer

$H = \left(\frac{23}{9}, -\frac{1}{9}, \frac{28}{9}\right)$, $\quad d(Q,\pi) = \frac{4}{3}$.

Exercise 16.

Unit normal $\widehat{\vec{\mathbf{n}}} = (\frac{2}{3}, -\frac{2}{3}, \frac{1}{3})$, plane contains $(3,3,3)$. (a) Hesse form and distance from O. (b) Integer Cartesian equation. (c) Distance from $(6,0,0)$.

(a) Check that $\widehat{\vec{\mathbf{n}}}$ is a unit vector: $\frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1$ ✓. In Hesse form $\widehat{\vec{\mathbf{n}}} \cdot (x,y,z) = p$, the value p is obtained from any point of the plane:

$$p = \widehat{\vec{\mathbf{n}}} \cdot (3,3,3) = 2 - 2 + 1 = 1 \geq 0.$$

So the given orientation is the correct one (pointing from O toward the plane), and

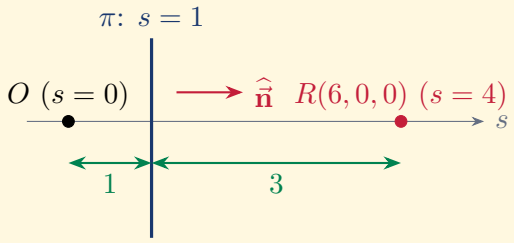
$$\boxed{\tfrac{2}{3}x - \tfrac{2}{3}y + \tfrac{1}{3}z = 1}, \qquad d(O,\pi) = 1.$$

(b) Multiply by 3: $2x - 2y + z = 3$, i.e. $\boxed{2x - 2y + z - 3 = 0}$.

(c) $d\big((6,0,0),\pi\big) = \dfrac{|12 - 0 + 0 - 3|}{\sqrt{4+4+1}} = \dfrac{9}{3} = 3.$

A one-dimensional picture

Measure every point by its coordinate along $\widehat{\vec{\mathbf{n}}}$, i.e. $s(P) = \widehat{\vec{\mathbf{n}}} \cdot \overrightarrow{OP}$. The plane is the level $s = 1$. The origin has $s = 0$ and $R(6,0,0)$ has $s = \frac{2}{3} \cdot 6 = 4$. Distances to the plane are differences of levels: $d(O,\pi) = |0 - 1| = 1$, $d(R,\pi) = |4 - 1| = 3$. Both points are on opposite sides ($0 < 1 < 4$).



Answer

$\frac{2}{3}x - \frac{2}{3}y + \frac{1}{3}z = 1$, $d(O,\pi) = 1$; $\quad 2x - 2y + z - 3 = 0$; $\quad d\big((6,0,0),\pi\big) = 3.$

Intersections

Exercise 17.

$\pi : 3x - y + z - 10 = 0$. Classify $L_1 = (2,-1,3) + t(1,2,-1)$, $L_2 = t(1,2,-1)$, $L_3 = (2,-1,3) + t(0,0,1)$. Why does $\vec{\mathbf{n}} \cdot \vec{\mathbf{d}} = 0$ alone not distinguish (a) from (b)?

$\vec{\mathbf{n}} = (3,-1,1)$. For each line, compute $\vec{\mathbf{n}} \cdot \vec{\mathbf{d}}$ and substitute.

(a) $\vec{\mathbf{d}} = (1,2,-1)$: $\vec{\mathbf{n}} \cdot \vec{\mathbf{d}} = 3 - 2 - 1 = 0$, so L_1 is parallel to π (disjoint or contained). Substitution:

$$3(2+t) - (-1+2t) + (3-t) - 10 = (6 + 1 + 3 - 10) + (3t - 2t - t) = 0.$$

The identity $0 = 0$ holds for every t: L_1 **is contained in** π. (Its starting point satisfies $6 + 1 + 3 - 10 = 0$.)

(b) Same $\vec{\mathbf{d}}$, so again $\vec{\mathbf{n}} \cdot \vec{\mathbf{d}} = 0$. Substitution:

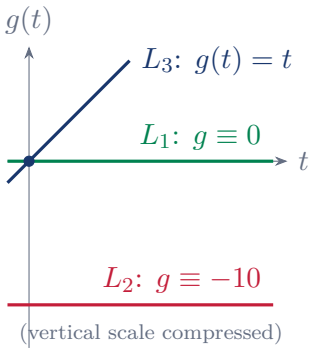
$$3t - 2t + (-t) - 10 = -10 = 0,$$

which is false for every t: L_2 **is parallel and disjoint**. (Its starting point O gives $-10 \neq 0$.)

(c) $\vec{\mathbf{d}} = (0,0,1)$: $\vec{\mathbf{n}} \cdot \vec{\mathbf{d}} = 1 \neq 0$, so exactly one intersection point. Substitution:

$$3(2) - (-1) + (3+t) - 10 = t = 0 \implies t = 0,$$

one point: $(2,-1,3)$, which is the starting point itself. Check: $6+1+3-10=0$ ✓.



$g(t)$ is the left-hand side of π's equation evaluated along the line.
The line meets π exactly where $g(t)=0$: everywhere, nowhere, or at one t.

Why $\vec{\mathbf{n}}\cdot\vec{\mathbf{d}}=0$ is not enough. L_1 and L_2 have the *same* direction vector, hence the same value $\vec{\mathbf{n}}\cdot\vec{\mathbf{d}}=0$. That number only tells that the lines are *parallel to the plane*. Whether the line lies *in* the plane depends on its *position*, i.e. on one point: $(2,-1,3)\in\pi$ but $(0,0,0)\notin\pi$. In the substitution, $\vec{\mathbf{n}}\cdot\vec{\mathbf{d}}$ is the coefficient of t (zero in both cases) and the starting point produces the constant term (0 vs. -10).

Answer

$L_1\subset\pi$; $L_2\parallel\pi$, disjoint; $L_3\cap\pi=\{(2,-1,3)\}$.

Exercise 18.

$L:(x,y,z)=(0,1,-2)+t(2,-1,1)$ and $\pi: x+y+z=5$. (a) Exactly one intersection point. (b) Find it and check.

(a) $\vec{\mathbf{n}}=(1,1,1)$, $\vec{\mathbf{d}}=(2,-1,1)$:

$$\vec{\mathbf{n}}\cdot\vec{\mathbf{d}}=2-1+1=2\neq 0,$$

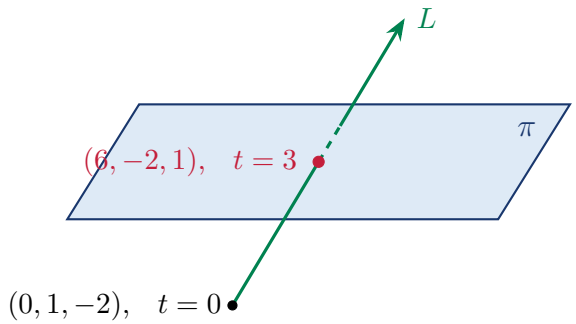
so L is not parallel to π and meets it in exactly one point (Exercise 4).

(b) Substitute $x=2t$, $y=1-t$, $z=-2+t$:

$$2t+(1-t)+(-2+t)=5 \iff 2t-1=5 \iff t=3.$$

Point: $(0,1,-2)+3(2,-1,1)=(6,-2,1)$.

Check on L: $t=3$ gives $(6,-2,1)$ ✓. Check on π: $6-2+1=5$ ✓.



Schematic. Along L, $x+y+z-5=2t-6$ changes sign at $t=3$: the starting point ($-6<0$) and, e.g., $t=4$ ($+2>0$) lie on opposite sides of π.

Answer

$\vec{\mathbf{n}}\cdot\vec{\mathbf{d}}=2\neq 0$; the unique intersection point is $(6,-2,1)$ (parameter $t=3$).