Chapter 5 - Exercises

Matrices : Definitions, special matrices, operations and properties

Collège Sainte-Anne | Fall 2026

How to use this series

COMPUTE apply a definition TRUE / FALSE decide, then justify THINK understand why DRAW make a figure PROVE write an argument

Difficulty: ⋆ direct application, ⋆⋆ standard, ⋆⋆⋆ subtle.
True/false rule. If a statement is true, give a short general argument. If it is false, give *one explicit counterexample*. Small 2×2 matrices with many zeros are usually the best counterexamples.

1 Reading and building matrices

Reminder

A matrix of size $m \times n$ has m rows and n columns. The entry a_{ij} is in **row i, column j**: rows always come first.

Exercise 1.1 COMPUTE ⋆

Let $M = \begin{bmatrix} 2 & -3 & 0 & 5 \\ 1 & 4 & -6 & 2 \\ 7 & 0 & 3 & -1 \end{bmatrix}$.

(a) What is the size of M? How many entries does it have?

(b) Give m_{14}, m_{31}, m_{23}. Does m_{42} exist?

(c) Compute the sum of the entries of row 2, then of column 3.

(d) Compute $m_{11} + m_{22} + m_{33}$.

Exercise 1.2 COMPUTE ⋆

Write explicitly each matrix.

(a) $A = [a_{ij}]$ of size 3×3 with $a_{ij} = (-1)^{i+j}$.

(b) $B = [b_{ij}]$ of size 2×4 with $b_{ij} = 2i - j$.

(c) $C = [c_{ij}]$ of size 3×3 with $c_{ij} = \min(i, j)$.

Exercise 1.3 THINK ⋆⋆ *Going backwards*

Find a formula $a_{ij} = f(i, j)$ that produces each matrix.

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

Exercise 1.4 THINK ⋆⋆

(a) A matrix has exactly 12 entries. List all its possible sizes. Can it be square?

(b) Same question for 16 entries.

Exercise 1.5 COMPUTE ⋆ *A school supply store*

A company has two stores. The matrix P gives the number of items in stock (rows: Laval, Montréal; columns: notebooks, pens, calculators):

$$P = \begin{bmatrix} 120 & 300 & 15 \\ 80 & 450 & 22 \end{bmatrix}.$$

(a) What does p_{23} represent? Which entry gives the number of pens in Laval?

(b) How many pens does the company have in total? Which entries did you add?

We will come back to this company in Section 5.

2 Special matrices and equality

Reminder

Square: $m = n$. *Diagonal*: $a_{ij} = 0$ if $i \neq j$. *Upper triangular*: $a_{ij} = 0$ if $i > j$. *Lower triangular*: $a_{ij} = 0$ if $i < j$. *Scalar*: kI_n. Two matrices are **equal** if they have the same size **and** the same entries.

Exercise 2.1 COMPUTE ⋆

For each matrix, list *all* the words that apply: square, zero, identity, diagonal, scalar, upper triangular, lower triangular.

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 2 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad G = \begin{bmatrix} 1 \end{bmatrix}$$

Exercise 2.2 COMPUTE ⋆

Find x, y and z such that $\begin{bmatrix} x+2y & 3 \\ -1 & x-y \end{bmatrix} = \begin{bmatrix} 7 & z^2-1 \\ -1 & 1 \end{bmatrix}$.

Exercise 2.3 THINK ⋆⋆ *When equality is impossible, or too easy*

(a) Is there a real x such that $\begin{bmatrix} x & x^2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$?

(b) Can we find a, b, c such that $\begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$?

(c) Find *all* the matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b & c \\ d & a \end{bmatrix}$.

Exercise 2.4 TRUE / FALSE ★★

(a) A 1×1 matrix is always diagonal.

(b) The square zero matrix $O_{n\times n}$ is a scalar matrix.

(c) A lower triangular matrix whose diagonal entries are all 0 is the zero matrix.

(d) The matrix $[a_{ij}]$ of size 3×3 with $a_{ij} = ij$ is upper triangular.

(e) A matrix that is both upper triangular and lower triangular is diagonal.

(f) $F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is the identity matrix of size 2×3.

Exercise 2.5 THINK ★★★ *Counting*

Consider all 2×2 matrices whose entries are 0 or 1.

(a) How many such matrices are there?

(b) How many of them are upper triangular? diagonal? scalar?

(c) How many are *both* upper and lower triangular? Compare with (b).

3 Addition and multiplication by a scalar

Reminder

$A + B$ is defined only if A and B have the same size; we add entry by entry. kA multiplies *every* entry by k. These operations follow the same rules as numbers (or vectors), so we can solve equations as usual.

Throughout this section,

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 4 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}.$$

Exercise 3.1 COMPUTE ★

Compute, if defined: (a) $A + B$ (b) $2A - 3B$ (c) $-A + \frac{1}{2}B$ (d) $A + C$ (e) $3C - C$.

Exercise 3.2 COMPUTE ★★ *Matrix equations*

(a) Find the matrix X such that $3X - 2A = B + X$.

(b) Find the matrices X and Y such that $X + Y = A$ and $X - Y = B$.

Hint: isolate X first, exactly as with numbers, and compute only at the end.

Exercise 3.3 THINK ★★

Let $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(a) Find r, s such that $rI_2 + sJ = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$.

(b) Show that no r, s satisfy $rI_2 + sJ = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$.

(c) Describe all the 2×2 matrices that can be written in the form $rI_2 + sJ$.

Exercise 3.4 TRUE / FALSE ⋆⋆ *A, B, C are matrices of the same size and k is a real number*

(a) If $A + B = A + C$, then $B = C$.

(b) If $kA = O$, then $k = 0$ or $A = O$.

(c) If $kA = kB$, then $A = B$.

(d) $A - B = B - A$ for all A and B.

(e) If $A + B = O$, then $B = -A$.

Keep your answers to (a) and (b) in mind: we will compare them with the matrix product in Section 6.

Exercise 3.5 PROVE ⋆⋆

Using entries, prove that $(r + s)A = rA + sA$ and that $A + (-A) = O$.

4 The transpose

Reminder

$(A^T)_{ij} = a_{ji}$: the rows of A become the columns of A^T. If A is $m \times n$, then A^T is $n \times m$. A is *symmetric* if $A^T = A$, and *antisymmetric* if $A^T = -A$.

Exercise 4.1 COMPUTE ⋆

With A and B from Section 3, compute A^T, B^T, $(A+2B)^T$ and A^T+2B^T. What do you notice?

Exercise 4.2 COMPUTE ⋆

(a) Find x, y, z so that $\begin{bmatrix} 2 & x+1 & 3 \\ 4 & 0 & y \\ z-1 & 5 & 1 \end{bmatrix}$ is symmetric.

(b) Find a, b, c, d, e, f so that $\begin{bmatrix} a & 2 & b \\ c & d & -4 \\ 1 & e & f \end{bmatrix}$ is antisymmetric.

Exercise 4.3 TRUE / FALSE ⋆⋆

(a) If A is 2×3, then $A + A^T$ is defined.

(b) A symmetric matrix is necessarily square.

(c) The diagonal entries of an antisymmetric matrix are all zero.

(d) If A is upper triangular, then A^T is upper triangular.

(e) The sum of two symmetric matrices is symmetric.

(f) Every diagonal matrix is symmetric.

(g) A matrix that is both symmetric and antisymmetric is the zero matrix.

Exercise 4.4 COMPUTE ⋆⋆

Write $A = \begin{bmatrix} 1 & 4 & -2 \\ 0 & 3 & 5 \\ 6 & 1 & 0 \end{bmatrix}$ as $A = S + K$, with S symmetric and K antisymmetric. Check your answer.

Exercise 4.5 PROVE ⋆⋆⋆ *The decomposition is unique*

Suppose $A = S+K = S'+K'$ with S, S' symmetric and K, K' antisymmetric. Show that $S = S'$ and $K = K'$. *Hint:* what can you say about the matrix $S - S' = K' - K$? Use Exercise 4.3(g).

Exercise 4.6 THINK ⋆⋆⋆ *How much freedom?*

To choose a symmetric 3×3 matrix, how many entries can you choose freely? What about an antisymmetric 3×3 matrix? Generalise to $n \times n$ and check that the two answers add up to n^2. How is this related to Exercise 4.4?

5 The matrix product: computing

Reminder

If A is $m \times n$ and B is $n \times p$, then AB is $m \times p$, and $(AB)_{ij} = (\text{row } i \text{ of } A) \cdot (\text{column } j \text{ of } B)$.
Column view: $A\vec{\mathbf{x}} = x_1\vec{\mathbf{a}}_1 + \cdots + x_n\vec{\mathbf{a}}_n$, a linear combination of the columns of A.

Exercise 5.1 COMPUTE ⋆ *Size first*

A is 3×2, B is 2×4, C is 4×4, D is 4×1, E is 1×3. Which products are defined? Give the size of the result when it is.

$$AB, \quad BA, \quad BC, \quad CD, \quad DE, \quad ED, \quad EA, \quad AE, \quad ABD, \quad C^2, \quad D^TD, \quad DD^T, \quad EAB.$$

Exercise 5.2 COMPUTE ⋆

Let $P = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 4 & -2 \end{bmatrix}$, $\vec{\mathbf{u}} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$.

(a) Compute PQ and QP. Do they have the same size?

(b) Compute $\vec{\mathbf{u}}^T\vec{\mathbf{v}}$ and $\vec{\mathbf{u}}\,\vec{\mathbf{v}}^T$. Which of the two is a dot product?

Exercise 5.3 COMPUTE ⋆⋆ *One entry at a time*

Let $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 1 \\ 5 & 2 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 1 & 4 \\ 2 & 5 & -1 \end{bmatrix}$. *Without computing the whole product*, find $(AB)_{23}$, $(AB)_{31}$ and $(BA)_{23}$. Which part of A do you need to compute the second row of AB?

Exercise 5.4 COMPUTE ⋆⋆ *The store, continued*

Go back to the stock matrix P of Exercise 1.5. The table below gives the purchase cost and the selling price of each item (in \$):

$$C = \begin{bmatrix} 2.00 & 3.50 \\ 0.60 & 1.25 \\ 11.00 & 18.00 \end{bmatrix} \qquad \text{(rows: notebook, pen, calculator; columns: cost, selling price).}$$

(a) Explain why PC is defined and give its size.

(b) Compute PC. What do the four entries represent?

(c) Deduce the profit each store makes if it sells its whole stock.

(d) Is CP defined? Would it have a meaning?

Exercise 5.5 COMPUTE ⋆ *Systems and matrices*

(a) Write the system $\begin{cases} x - 2y + 3z = 4 \\ 2x + z = -1 \\ y - z = 0 \end{cases}$ in the form $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

(b) Write the system of equations that corresponds to $\begin{bmatrix} 3 & -1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$. How many equations and unknowns are there? Is $(x, y) = (2, 1)$ a solution? Show that this system has no solution.

Exercise 5.6 THINK ⋆⋆ *The column view*

Let $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$.

(a) Compute $A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. What do you get?

(b) Using the column view (no row · column computation), find a vector $\vec{\mathbf{x}}$ such that $A\vec{\mathbf{x}} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

(c) Compute $A \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$. Deduce that $A\vec{\mathbf{x}} = A\vec{\mathbf{y}}$ does *not* imply $\vec{\mathbf{x}} = \vec{\mathbf{y}}$.

Exercise 5.7 PROVE ⋆⋆⋆

Let A and B be two $m \times n$ matrices.

(a) Show that $A\vec{\mathbf{e}}_j$ is the j-th column of A, where $\vec{\mathbf{e}}_j$ is the column with a 1 in position j and 0 elsewhere.

(b) Deduce that if $A\vec{\mathbf{x}} = B\vec{\mathbf{x}}$ for *every* $\vec{\mathbf{x}} \in \mathbb{R}^n$, then $A = B$.

6 The matrix product: properties and traps

What changes compared with numbers

In general $AB \neq BA$. Also, $AB = O$ does not imply that $A = O$ or $B = O$, and $AB = AC$ does not imply $B = C$. Moreover, $(AB)^T = B^T A^T$. Associativity, distributivity (*keeping the order*) and $IA = AI = A$ remain true.

Exercise 6.1 TRUE / FALSE ⋆⋆ *Matrices of sizes for which the operations are defined*

(a) If AB and BA are both defined, then A and B are square.

(b) If AB and BA are both defined and have the same size, then A and B are square of the same order.

(c) If $AB = O$, then $A = O$ or $B = O$.

(d) If $A^2 = O$, then $A = O$.

(e) $(AB)^T = A^T B^T$.

(f) If A has a row of zeros, then AB has a row of zeros.

(g) If A has a column of zeros, then AB has a column of zeros.

(h) If B has a column of zeros, then AB has a column of zeros.

(i) $(A - I)(A + I) = A^2 - I$ for every square matrix A.

(j) For every matrix A, the product $A^T A$ is defined.

Exercise 6.2 COMPUTE ⋆⋆ *Who commutes?*

(a) Find all matrices $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $DB = BD$, where $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

(b) Explain why every matrix commutes with $3I_2$.

Exercise 6.3 THINK ⋆⋆ *Equations that behave strangely*

(a) Check that $A = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$ satisfies $A^2 = O$.

(b) For real numbers, $x^2 = 1$ has exactly two solutions. Check that $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ both satisfy $X^2 = I_2$. Then show that $X_t = \begin{bmatrix} 1 & t \\ 0 & -1 \end{bmatrix}$ satisfies $X_t^2 = I_2$ for every real t. How many solutions does $X^2 = I_2$ have?

(c) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Find two matrices $B \neq C$ such that $AB = AC$. *Hint:* $AB = AC \iff A(B - C) = O$.

Exercise 6.4 COMPUTE ⋆⋆ *Expanding correctly*

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

(a) Compute $(A+B)^2$ directly.

(b) Compute $A^2 + 2AB + B^2$, then $A^2 + AB + BA + B^2$. Which one equals $(A+B)^2$? Why?

(c) Show that for every square matrix A, $(A+I)^3 = A^3 + 3A^2 + 3A + I$. Why is this correct even though the matrix product is not commutative?

Exercise 6.5 COMPUTE ⋆⋆ *AA^T and A^TA*

Let $A = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 1 & -1 \end{bmatrix}$.

(a) Compute AA^T and A^TA. Give their sizes. What do you notice about these two matrices?

(b) Compute their traces and the sum of the squares of all the entries of A.

Exercise 6.6 PROVE ⋆⋆

(a) Show that AA^T and A^TA are always symmetric, using $(XY)^T = Y^TX^T$.

(b) Show that $(ABC)^T = C^TB^TA^T$.

(c) Show that if row i of A contains only zeros, then row i of AB contains only zeros.

7 Matrices as transformations of the plane

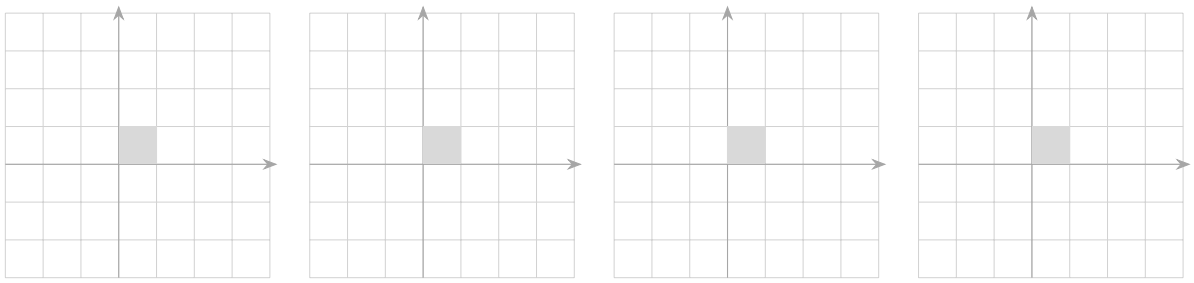
Reminder

The 2×2 matrix A sends $\vec{\mathbf{i}} = (1,0)$ to its **first column** and $\vec{\mathbf{j}} = (0,1)$ to its **second column**. The unit square becomes the parallelogram spanned by these two columns. The product AB means "apply B first, then A".

Exercise 7.1 DRAW ⋆

For each matrix, draw the image of the unit square (on a separate grid for each) and describe the transformation in words.

(a) $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$



Exercise 7.2 COMPUTE ⋆ *From images to matrix*

Find the matrix A that sends $\vec{\mathbf{i}}$ to $(2,1)$ and $\vec{\mathbf{j}}$ to $(-1,3)$. Then find the images of $(1,1)$ and of $(3,-2)$.

Exercise 7.3 THINK ⋆⋆ *Order matters*

Let F be the reflection across the x-axis and R the rotation by $90°$ counterclockwise.

(a) Write F and R by looking at the images of $\vec{\mathbf{i}}$ and $\vec{\mathbf{j}}$.

(b) Compute FR and RF. Is the order important?

(c) Follow the point $(1,0)$: first through R then F, then the other way round. Check that the results agree with (b). Identify FR and RF geometrically.

Exercise 7.4 THINK ⋆⋆ *A projection*

Let $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(a) Describe what P does to a point (x,y). Draw the image of the unit square.

(b) Compute P^2. Explain the result geometrically, without computing.

(c) Explain geometrically why P cannot be invertible, then check it with $ad-bc$.

Exercise 7.5 THINK ⋆⋆⋆ *When the square collapses*

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

(a) Draw the images of $\vec{\mathbf{i}}$ and $\vec{\mathbf{j}}$. What is the image of the unit square?

(b) Show that every vector $A\vec{\mathbf{x}}$ lies on one fixed line through the origin.

(c) Compute $ad-bc$. How is it related to (a)?

Exercise 7.6 PROVE ⋆⋆⋆ *Rotations and trigonometry*

The rotation by an angle θ counterclockwise is $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

(a) Check that $R_{90°}$ is the matrix R of Exercise 7.3.

(b) Rotating by β then by α is rotating by $\alpha+\beta$, so $R_\alpha R_\beta = R_{\alpha+\beta}$. Compute $R_\alpha R_\beta$ and deduce the formulas for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$.

(c) Do two rotations commute? Justify with (b).

(d) Show that $R_\theta^T R_\theta = I_2$. What is the inverse of a rotation?

8 Powers and trace

Reminder

$A^k = AA\cdots A$ (k factors) for a square matrix A, with $A^0 = I$. $\operatorname{tr}(A) = a_{11} + \cdots + a_{nn}$, and $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Exercise 8.1 COMPUTE ⋆⋆ *Guess, then prove*

Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.

(a) Compute A^2 and A^3.

(b) Conjecture a formula for A^n, then prove it by induction.

(c) Use it to write A^{10} without any further multiplication.

Exercise 8.2 COMPUTE ⋆⋆ *A shortcut*

Let $J = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

(a) Show that $J^2 = 2J$.

(b) Deduce J^3 and J^4 from (a) *without multiplying matrices*, then give a formula for J^n.

Exercise 8.3 COMPUTE ⋆⋆

Let $N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Compute N^2, N^3 and N^4, and describe how the 1's move each time. What is the smallest k with $N^k = O$?

Exercise 8.4 TRUE / FALSE ⋆⋆ *A, B, C are square matrices of the same order*

(a) $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$.

(b) $\operatorname{tr}(AB) = \operatorname{tr}(A)\operatorname{tr}(B)$.

(c) $\operatorname{tr}(A^T) = \operatorname{tr}(A)$.

(d) $\operatorname{tr}(ABC) = \operatorname{tr}(CAB)$.

(e) $\operatorname{tr}(ABC) = \operatorname{tr}(BAC)$.

(f) If $\operatorname{tr}(A) = 0$, then $A = O$.

Exercise 8.5 PROVE ⋆⋆⋆

(a) Show that $\operatorname{tr}(A^T A)$ is the sum of the squares of all the entries of A. (Check it with Exercise 6.5.)

(b) Deduce that if $A^T A = O$, then $A = O$.

9 The inverse of a matrix

Reminder

A^{-1} is the matrix such that $AA^{-1} = A^{-1}A = I$. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad - bc \neq 0$: $A^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. If $ad - bc = 0$, then A is not invertible. Also, $(AB)^{-1} = B^{-1}A^{-1}$.

Exercise 9.1 COMPUTE ⋆

For each matrix, find its inverse or explain why it does not exist. Check one of your answers by multiplying.

(a) $\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Exercise 9.2 COMPUTE ⋆

Use the inverse found in Exercise 9.1(a) to solve $\begin{cases} 5x + 2y = 1 \\ 7x + 3y = 2 \end{cases}$. Check your solution.

Exercise 9.3 COMPUTE ⋆⋆

For which values of k is the matrix invertible? (a) $\begin{bmatrix} k & 3 \\ k & k \end{bmatrix}$ (b) $\begin{bmatrix} 1-k & 2 \\ 3 & 2-k \end{bmatrix}$

Exercise 9.4 THINK ⋆⋆ *Left or right?*

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$.

(a) Solve $AX = B$. *On which side must you multiply by A^{-1}?*

(b) Solve $XA = B$.

(c) Compare the two answers. Why is it meaningless to write $X = \dfrac{B}{A}$?

Exercise 9.5 THINK ⋆⋆ *An inverse without a formula*

(a) Suppose that a square matrix A satisfies $A^2 - 3A + 2I = O$. Show that A is invertible and that $A^{-1} = \frac{1}{2}(3I - A)$. *Hint:* write the relation as $A \cdot (\dots) = I$.

(b) Check that $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ satisfies this relation, then compare $\frac{1}{2}(3I - A)$ with the 2×2 formula.

Exercise 9.6 THINK ⋆⋆⋆

Let $A = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$ (so $A^2 = O$, see Exercise 6.3).

(a) Show in two different ways that A is not invertible: (i) with $ad - bc$; (ii) without any computation, by assuming that A^{-1} exists and multiplying $A^2 = O$ by A^{-1}.

(b) Show that $I - A$ is invertible and that $(I - A)^{-1} = I + A$.

Exercise 9.7 TRUE / FALSE ⋆⋆ *A and B are square matrices of the same order*

(a) If A and B are invertible, then $A + B$ is invertible.

(b) If A and B are invertible, then AB is invertible and $(AB)^{-1} = A^{-1}B^{-1}$.

(c) $(A + B)^{-1} = A^{-1} + B^{-1}$ whenever all these inverses exist.

(d) If A is invertible and $AB = AC$, then $B = C$.

(e) If A is invertible and $AB = O$, then $B = O$.

(f) A 2×2 matrix with a row of zeros is never invertible.

(g) If A is invertible, then A^T is invertible.

10 Synthesis

Exercise 10.1 THINK ⋆⋆ *Spot the error*

Each argument below contains an error. Find it and give a counterexample to the conclusion.

(a) "$A^2 = A$, so $A^2 - A = O$, so $A(A - I) = O$. Therefore $A = O$ or $A = I$."

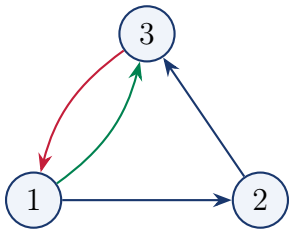
(b) "$AB = AC$. Simplifying by A on both sides, $B = C$."

(c) "Since $(A + B)^T = A^T + B^T$, by the same reasoning $(AB)^T = A^T B^T$."

(d) "$(AB)^2 = ABAB = AABB = A^2B^2$."

Exercise 10.2 COMPUTE ⋆⋆ *Counting flights*

Three cities are linked by one-way flights: $1 \to 2$, $2 \to 3$, $3 \to 1$ and $1 \to 3$. We define $M = [m_{ij}]$ by $m_{ij} = 1$ if there is a direct flight from i to j, and $m_{ij} = 0$ otherwise.



(a) Write M. Is it symmetric? Why not?

(b) Compute M^2. Explain why $(M^2)_{ij} = m_{i1}m_{1j} + m_{i2}m_{2j} + m_{i3}m_{3j}$ counts the routes from i to j with exactly two flights. List the routes counted by $(M^2)_{13}$.

(c) Compute $M + M^2$. Starting from city 2, which cities can you reach with at most two flights? Can you come back to 2?

Exercise 10.3 PROVE ⋆⋆⋆

Let A and B be symmetric matrices of the same order. Show that AB is symmetric if and only if $AB = BA$.

Exercise 10.4 PROVE ⋆⋆⋆ *Matrices that commute with everything*

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix that commutes with *every* 2×2 matrix.

(a) Use $E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ to show that $b = c = 0$.

(b) Use $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to show that $a = d$.

(c) Conclude: which matrices commute with all 2×2 matrices?