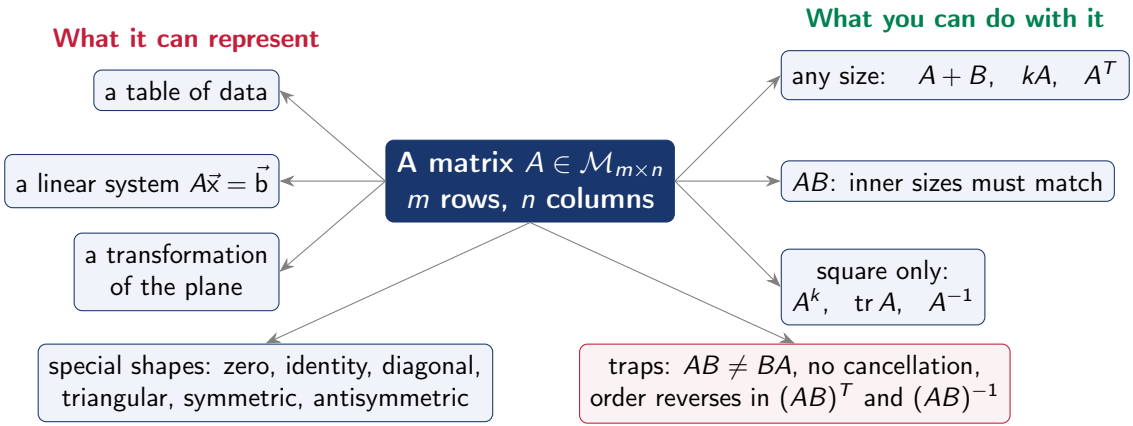
Chapter 5 – Matrices

Summary: key ideas and examples

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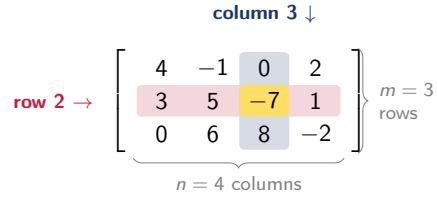
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The chapter at a glance



1. Anatomy of a matrix

Rows first, columns second



size 3×4 (12 entries) entry $a_{23} = -7$

Notation

$A = [a_{ij}] \in \mathcal{M}_{m \times n}$: a_{ij} is in **row** i, **column** j.

Vocabulary

- $1 \times n$: **row matrix**; $m \times 1$: **column matrix** (these are vectors).
- $n \times n$: **square**; its **main diagonal** is $a_{11}, \ldots, a_{nn}$.
- A rule $a_{ij} = f(i, j)$ can define a matrix.

$a_{ij} = 2i - j$, size 2×3

$$\begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \end{bmatrix}$$

2. Special square matrices: look at where the zeros are

shaded = may be nonzero, white = must be 0

0	0	0
0	0	0
0	0	0

zero O
all $a_{ij} = 0$

1	0	0
0	1	0
0	0	1

identity I_3
1 on diagonal

k	0	0
0	k	0
0	0	k

scalar kI_3
same k on diagonal

$*$	0	0
0	$*$	0
0	0	$*$

diagonal
$i \neq j \Rightarrow a_{ij} = 0$

$*$	$*$	$*$
0	$*$	$*$
0	0	$*$

upper triangular
$i > j \Rightarrow a_{ij} = 0$

$*$	0	0
$*$	$*$	0
$*$	$*$	$*$

lower triangular
$i < j \Rightarrow a_{ij} = 0$

Classify

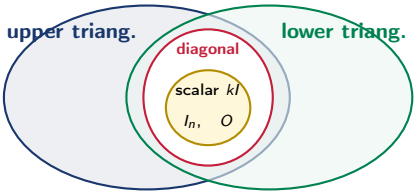
$\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ diagonal (so also upper and lower triangular), not scalar.

$\begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$ upper triangular only: $a_{12} = 1 \neq 0$.

Memory trick

The name says where the **nonzero** part lives: *upper* triangular = triangle *above* (and on) the diagonal.

Inclusions (square matrices of order n)



upper **and** lower triangular $\iff$ diagonal

Equality $A = B$

same size and $a_{ij} = b_{ij}$ for all i, j.

Find x, y, z

$$\begin{bmatrix} 2x & y+1 \\ 0 & z \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 0 & -2 \end{bmatrix}$$

Compare position by position: $2x = 6$, $y + 1 = 4$, $z = -2$, so $x = 3$, $y = 3$, $z = -2$.

Same numbers $\neq$ same matrix

$\begin{bmatrix} 1 & 2 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (sizes 1×2 vs 2×1).

3. Addition and scalar multiplication: entry by entry

Each position only talks to the *same* position (same size required!)

$$\begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 1 & 3 \end{bmatrix}$$

$1+2,\ -2+5,\ 0+1,\ 4-1$

$$3\begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 0 & 12 \end{bmatrix}$$

every entry $\times 3$

Same rules as numbers and vectors

$A+B=B+A,\quad (A+B)+C=A+(B+C),$
$A+O=A,\quad A+(-A)=O,$
$r(A+B)=rA+rB,\quad (r+s)A=rA+sA,$
$r(sA)=(rs)A.$

Solve $3X+A=B$ like a number equation

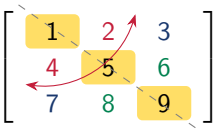
$$A=\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix},\quad B=\begin{bmatrix} 7 & 3 \\ -1 & 5 \end{bmatrix}:$$

$$X=\tfrac{1}{3}(B-A)=\tfrac{1}{3}\begin{bmatrix} 6 & 3 \\ -3 & 6 \end{bmatrix}=\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}.$$

4. The transpose: rows become columns

$(A^T)_{ij} = a_{ji}$ **size** $m \times n \;\rightarrow\; n \times m$

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 4 & -1 \end{bmatrix} \longrightarrow A^T = \begin{bmatrix} 1 & 0 \\ -2 & 4 \\ 3 & -1 \end{bmatrix}$$



square case: **mirror** across the diagonal; diagonal stays put

Rules

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(kA)^T = kA^T$
4. $(AB)^T = B^T A^T$ **order reversed**

Quick ones

$$\begin{bmatrix} 5 & -1 & 2 \end{bmatrix}^T = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}, \qquad I_n^T = I_n.$$

The transpose of an upper triangular matrix is lower triangular.

Symmetric and antisymmetric matrices

Symmetric: $A^T = A$

$a_{ji} = a_{ij}$: mirror image across the diagonal.

$$\begin{bmatrix} 1 & 4 & 0 \\ 4 & -3 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$

Antisymmetric: $A^T = -A$

$a_{ji} = -a_{ij}$, and so the diagonal is 0 ($a_{ii} = -a_{ii}$).

$$\begin{bmatrix} 0 & 4 & -1 \\ -4 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$$

Every square matrix = symmetric + antisymmetric: $S = \frac{1}{2}(A + A^T)$, $K = \frac{1}{2}(A - A^T)$

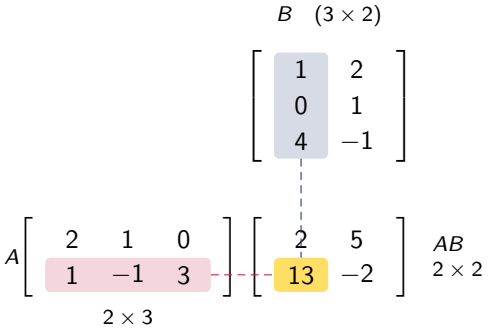
$A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$, $A^T = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \implies S = \frac{1}{2}\begin{bmatrix} 6 & 6 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 2 \end{bmatrix}$, $K = \frac{1}{2}\begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$. Check: $S + K = A$ ✓

Warning

Both notions only make sense for **square** matrices (A and A^T must have the same size).

5. The product AB: row · column

Place B above, A to the left: each entry is at a crossing



$$(AB)_{21} = (1)(1) + (-1)(0) + (3)(4) = 13$$

Definition

$A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{n\times p}$:

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$=$ (row i of A) $\cdot$ (column j of B).

Size rule

$$(m \times n)\,(n \times p) \;\rightarrow\; m \times p$$

must match

Like dominoes: the touching ends must be equal; the outer ends give the size.

A: 2×3, B: 3×4, D: 3×3

Product	Inner	Result
AB	$3 = 3$	2×4
BA	$4 \neq 2$	undefined
AD	$3 = 3$	2×3
DA	$3 \neq 2$	undefined
A^2	$3 \neq 2$	undefined
$A^T A$	$2 = 2$	3×3

Big idea

A^2 exists only for **square** A.

Row × column vs column × row

$$\begin{bmatrix} 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \end{bmatrix} \quad (1 \times 1)$$

$$\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 3 \\ 8 & -4 & 12 \\ 4 & -2 & 6 \end{bmatrix} \quad (3 \times 3)$$

Same two factors, swapped: a number vs a 3×3 matrix!

The identity changes nothing

$I_m A = A = A I_n$ for $A \in \mathcal{M}_{m \times n}$.

The column view: $A\vec{x}$ is a combination of the columns of A

Key formula

If $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}$ (columns), then

$$A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n.$$

Example

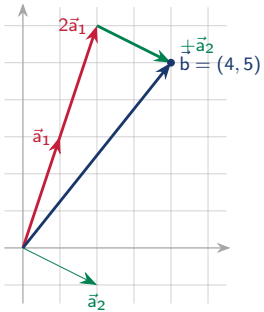
$$\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

A linear system is one matrix equation $A\vec{x} = \vec{b}$

$$\begin{cases} x + 2y = 4 \\ 3x - y = 5 \end{cases} \iff \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

Solving = finding the combination of columns that reaches $\vec{b}$. Here: $x = 2$, $y = 1$ (picture $\rightarrow$).

$2\vec{a}_1 + 1\vec{a}_2$ reaches $(4, 5)$



Properties of the product

What still works

- $(AB)C = A(BC)$ — associativity
- $A(B+C) = AB + AC$, $(A+B)C = AC + BC$ — distributivity
- $k(AB) = (kA)B = A(kB)$
- $IA = AI = A$, $\quad OA = O$, $\quad AO = O$

Keep the side!

$A(B+C) = AB + AC$: A stays **on the left**. Writing $BA + CA$ is wrong.

What fails (next slide)

- commutativity: $AB \neq BA$ in general
- $AB = O \not\Rightarrow A = O$ or $B = O$
- $AB = AC \not\Rightarrow B = C$
- $(A+B)^2 \neq A^2 + 2AB + B^2$ in general

Expanding correctly

$(A+B)^2 = A^2 + AB + BA + B^2$
$(A+I)^2 = A^2 + 2A + I$ ✓ (since $AI = IA$)

Traps of the product, with counterexamples

1. $AB \neq BA$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}:$$

$$AP = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}, \qquad PA = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}.$$

P on the right swaps the **columns**; on the left it swaps the **rows**.

2. Product zero, factors nonzero

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

3. No cancellation

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = I_2, \quad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}:$$

$$AB = A = AC \quad \text{but} \quad B \neq C.$$

Cancelling A is allowed only when A is **invertible**.

4. Squares of sums

With A, P from trap 1:

$$AP + PA = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} \neq 2AP = \begin{bmatrix} 4 & 2 \\ 8 & 6 \end{bmatrix},$$

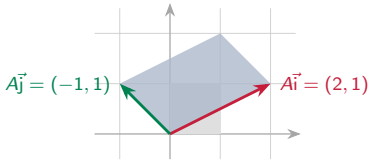
so $(A+P)^2 \neq A^2 + 2AP + P^2$.

6. A 2×2 matrix moves the plane: read the columns

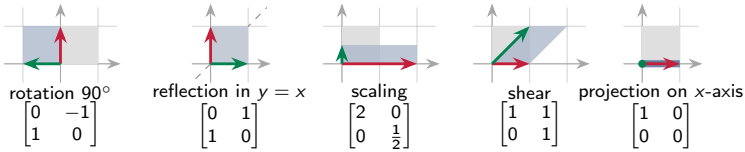
The columns are the images of $\vec{\imath}$ and $\vec{\jmath}$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$: $A\vec{\imath} = \begin{bmatrix} a \\ c \end{bmatrix}$, $A\vec{\jmath} = \begin{bmatrix} b \\ d \end{bmatrix}$.
The unit square goes to the parallelogram built on the two columns.

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$



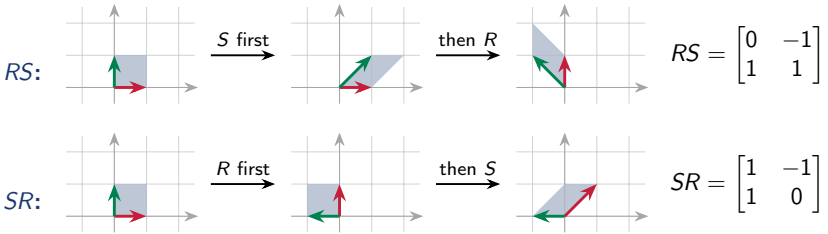
A small catalogue (grey: unit square, blue: image, red $= A\vec{\imath}$, green $= A\vec{\jmath}$)



The product = doing one transformation after the other

$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (shear), $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (rotation 90°). In $RS\vec{x} = R(S\vec{x})$, the matrix **on the right acts first**.

Two orders, two different results



Non-commutativity, seen

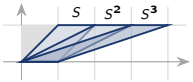
Different shapes at the end, so $RS \neq SR$. Check: the columns of RS are exactly where the red and green arrows land.

7. Powers of a square matrix

$$A^k = \underbrace{A\,A\cdots A}_{k}, \qquad A^0 = I_n$$

$A^pA^q = A^{p+q}$, $(A^p)^q = A^{pq}$, but in general $(AB)^k \neq A^kB^k$.

Shear powers: $S^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$



shearing n times = one shear n times as strong

Diagonal: power the diagonal

$$\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}^k = \begin{bmatrix} 2^k & 0 \\ 0 & (-1)^k \end{bmatrix}$$

e.g. $k = 5$: $\begin{bmatrix} 32 & 0 \\ 0 & -1 \end{bmatrix}$

Nilpotent: $N \neq O$ **but** $N^3 = O$

$$N = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix},$$

$$N^2 = \begin{bmatrix} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



nonzero entries are pushed away from the diagonal until nothing is left

8. The trace

Trace of a square matrix

$\operatorname{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$ (sum of the diagonal).

Visualisation

$$\begin{bmatrix} \mathbf{4} & -1 & 7 \\ 2 & \mathbf{5} & 0 \\ 8 & 3 & \mathbf{-6} \end{bmatrix} \quad \operatorname{tr} = 4 + 5 - 6 = 3$$

Properties

$\operatorname{tr}(A+B) = \operatorname{tr} A + \operatorname{tr} B$, $\operatorname{tr}(kA) = k \operatorname{tr} A$,
$\operatorname{tr}(A^T) = \operatorname{tr} A$, $\operatorname{tr}(I_n) = n$,
$\operatorname{tr}(AB) = \operatorname{tr}(BA)$ **even though** $AB \neq BA$.

$\operatorname{tr}(AB) = \operatorname{tr}(BA)$, even with different sizes!

With $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 4 & -1 \end{bmatrix}$ (slide 5):

$$AB = \begin{bmatrix} 2 & 5 \\ 13 & -2 \end{bmatrix} \quad (2 \times 2), \qquad \operatorname{tr}(AB) = 0,$$

$$BA = \begin{bmatrix} 4 & -1 & 6 \\ 1 & -1 & 3 \\ 7 & 5 & -3 \end{bmatrix} \quad (3 \times 3), \qquad \operatorname{tr}(BA) = 0.$$

Warning

$\operatorname{tr}(AB) \neq \operatorname{tr}(A)\operatorname{tr}(B)$ in general: $\operatorname{tr}(I_2 I_2) = 2$ but $\operatorname{tr}(I_2)\operatorname{tr}(I_2) = 4$.

9. Invertible matrices

Inverse

$A \in \mathcal{M}_n$ is **invertible** if some B satisfies $AB = BA = I_n$. Then $B = A^{-1}$ (unique).

2×2 formula

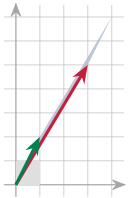
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det A = ad - bc.$$

$$\det A \neq 0: \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

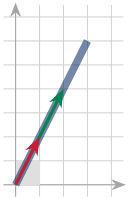
swap the diagonal, change the sign of the other two, divide. $\det A = 0$: **not** invertible.

Never write $\frac{1}{A}$ or $\frac{B}{A}$.

Why $\det A = 0$ means no inverse



$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$, $\det = 1$
still a parallelogram: can be undone



$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $\det = 0$
crushed onto a line: cannot be undone

Preview: $|ad - bc|$ is the area of the image of the unit square.

Solve $3x + y = 5,\quad 5x + 2y = 9$

$A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$, $\det A = 6 - 5 = 1$:

$$A^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}.$$

Multiply $A\vec{x} = \vec{b}$ on the **left** by A^{-1}:

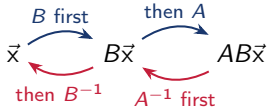
$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Check: $3 + 2 = 5$ ✓, $5 + 4 = 9$ ✓

Rules (A, B invertible)

$(A^{-1})^{-1} = A$, $(kA)^{-1} = \frac{1}{k}A^{-1}$, $(A^T)^{-1} = (A^{-1})^T$

Undo in the reverse order



$(AB)^{-1} = B^{-1}A^{-1}$, $(AB)^T = B^T A^T$

Socks and shoes

Socks go on first, shoes second. To undo: shoes come off first. The last operation done is the first one undone.

Cheat sheet

Operation	Defined when	Size of result	How to compute
$A + B$	same size	$m \times n$	add entry by entry
kA	always	$m \times n$	multiply every entry by k
A^T	always	$n \times m$	$(A^T)_{ij} = a_{ji}$ (rows $\to$ columns)
AB	A: $m \times n$, B: $n \times p$	$m \times p$	$(AB)_{ij} =$ row i of $A \cdot$ column j of B
A^k	A square	$n \times n$	$A \cdot A \cdots A$, $\quad A^0 = I_n$
$\operatorname{tr} A$	A square	a number	sum of the diagonal
A^{-1} (2×2)	$ad - bc \neq 0$	2×2	$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Always true

associativity, distributivity, $IA = AI = A$, $\operatorname{tr}(AB) = \operatorname{tr}(BA)$

Order matters

$AB \neq BA$; $\quad (AB)^T = B^T A^T$; $\quad (AB)^{-1} = B^{-1}A^{-1}$

Self-check: true or false?

Answer, then justify in one line

1. If A is 2×3 and B is 3×2, then AB and BA are both defined.
2. Every diagonal matrix is symmetric.
3. If $AB = O$, then $A = O$ or $B = O$.
4. $(AB)^T = A^T B^T$ for all square A, B.
5. $\operatorname{tr}(AB) = \operatorname{tr}(A)\ \operatorname{tr}(B)$.
6. $\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$ is invertible.
7. For every square matrix A, the matrix $A + A^T$ is symmetric.
8. The second column of A is the image of $\vec{\jmath}$ under A.

Self-check: answers

1. **True**: AB is 2×2 and BA is 3×3 (but they cannot be equal).
2. **True**: the entries off the diagonal are all 0, so $a_{ij} = a_{ji}$.
3. **False**: $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} = O$ with both factors nonzero.
4. **False**: the order reverses, $(AB)^T = B^T A^T$, and $A^T B^T \neq B^T A^T$ in general.
5. **False**: $\operatorname{tr}(I_2 I_2) = 2$, while $\operatorname{tr}(I_2)\operatorname{tr}(I_2) = 4$.
6. **False**: $ad - bc = 4 - 4 = 0$; its columns are parallel, so it crushes the plane onto a line.
7. **True**: $(A + A^T)^T = A^T + (A^T)^T = A^T + A$.
8. **True**: $A\vec{\mathrm{j}} = A \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ picks out column 2.