Chapter 5 – Summary

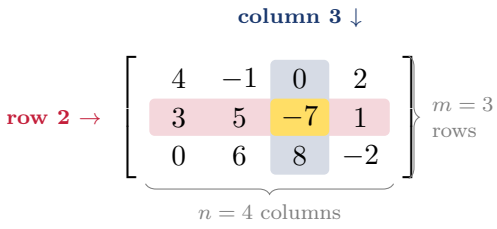
Matrices: key ideas and examples

Collège Sainte-Anne | Fall 2026

How to use this summary

Each section gives the definitions, one or two worked examples, and a picture. The **red boxes** are the traps that cost the most points on exams. Finish with the self-check on the last page *before* reading the answers.

1 What is a matrix?



size 3×4 (12 entries), $\quad a_{23} = -7$

Matrix of size $m \times n$

A rectangular array $A = [a_{ij}] \in \mathcal{M}_{m\times n}$ with m rows and n columns. The entry a_{ij} is in **row i**, **column j**: **rows first, columns second.**

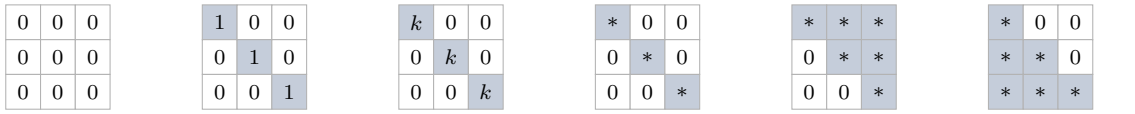
Vocabulary. A $1 \times n$ matrix is a **row matrix**, an $m \times 1$ matrix a **column matrix** (both play the role of vectors). An $n \times n$ matrix is **square**; its **main diagonal** is $a_{11}, a_{22}, \ldots, a_{nn}$. Entry a_{ij} lies *on* the diagonal if $i = j$, *above* if $i < j$, *below* if $i > j$.

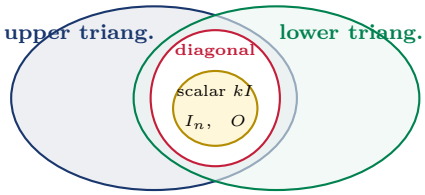
A matrix defined by a rule: $a_{ij} = 2i - j$, size 2×3

Substitute each position: $a_{11} = 2 - 1 = 1$, $a_{12} = 2 - 2 = 0$, $a_{13} = 2 - 3 = -1$, $a_{21} = 4 - 1 = 3$, ... $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \end{bmatrix}$.

2 Special matrices and equality

Special square matrices: shaded = may be nonzero, white = must be 0





upper **and** lower triangular $\iff$ diagonal

Classify

$\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$: diagonal, hence upper *and* lower triangular; not scalar ($2 \neq -3$).

$\begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$: upper triangular only, since $a_{12} = 1 \neq 0$.

Memory trick: the name says where the *nonzero* part lives; *upper* triangular = triangle *above* (and on) the diagonal.

Equality

$A = B$ means: **same size** and $a_{ij} = b_{ij}$ for every i, j. A matrix equation is a *system*, one equation per entry.

Find x, y, z

$\begin{bmatrix} 2x & y+1 \\ 0 & z \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 0 & -2 \end{bmatrix}$ gives $2x = 6$, $y+1 = 4$, $z = -2$: $x = 3$, $y = 3$, $z = -2$.

Same numbers ≠ same matrix

$\begin{bmatrix} 1 & 2 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 2 \end{bmatrix}$: sizes 1×2 and 2×1.

3 Addition and multiplication by a scalar

Entry by entry: each position only combines with the *same* position (same size required)

$$\begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 1 & 3 \end{bmatrix} \qquad 3\begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 0 & 12 \end{bmatrix}$$

$1+2,\ -2+5,\ 0+1,\ 4-1$ — every entry $\times 3$

Same rules as numbers and vectors

$A+B = B+A$, $(A+B)+C = A+(B+C)$,
$A+O = A$, $A+(-A) = O$,
$r(A+B) = rA+rB$, $(r+s)A = rA+sA$,
$r(sA) = (rs)A$, $1A = A$, $0A = O$.

Each one is proved *entry by entry*, using the same rule for real numbers.

Solve $3X + A = B$ like a number equation

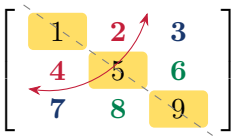
$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 7 & 3 \\ -1 & 5 \end{bmatrix}$. Isolate X first, compute at the end:

$$X = \tfrac{1}{3}(B - A) = \tfrac{1}{3}\begin{bmatrix} 6 & 3 \\ -3 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}.$$

Check: $3X + A = \begin{bmatrix} 6 & 3 \\ -3 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = B$ ✓

4 The transpose

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 4 & -1 \end{bmatrix} \longrightarrow A^T = \begin{bmatrix} 1 & 0 \\ -2 & 4 \\ 3 & -1 \end{bmatrix}$$



square case: a **mirror** across the diagonal; the diagonal stays put, a_{ij} and a_{ji} swap

Transpose

$A \in \mathcal{M}_{m\times n}$ gives $A^T \in \mathcal{M}_{n\times m}$ with $(A^T)_{ij} = a_{ji}$: the rows of A become the columns of A^T.

Rules

$(A^T)^T = A$, $(A+B)^T = A^T + B^T$, $(kA)^T = kA^T$, $(AB)^T = B^T A^T$ **(order reversed!)**

Symmetric: $A^T = A$, i.e. $a_{ji} = a_{ij}$

$$\begin{bmatrix} 1 & 4 & 0 \\ 4 & -3 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$ mirror image

Antisymmetric: $A^T = -A$, i.e. $a_{ji} = -a_{ij}$

$$\begin{bmatrix} 0 & 4 & -1 \\ -4 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$$ zero diagonal

Every square matrix = symmetric + antisymmetric: $S = \frac{1}{2}(A + A^T)$, $K = \frac{1}{2}(A - A^T)$

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix},\quad A^T = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \Longrightarrow S = \tfrac{1}{2}\begin{bmatrix} 6 & 6 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 2 \end{bmatrix},\quad K = \tfrac{1}{2}\begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}.$$

Check: $S + K = A$ ✓. Each entry of S is the *average* of a_{ij} and its mirror a_{ji}.

Warning

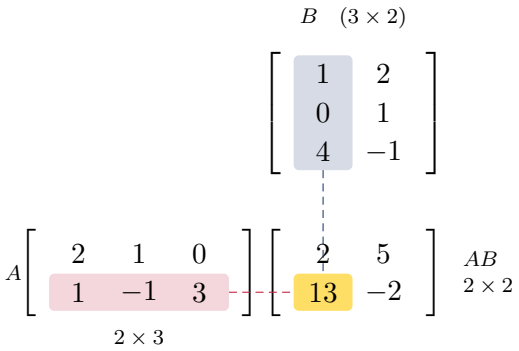
Symmetric and antisymmetric only make sense for **square** matrices (A and A^T must have the same size).

5 The matrix product

Product AB

If $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{n\times p}$, then $AB \in \mathcal{M}_{m\times p}$ with

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj} = (\text{row } i \text{ of } A)\cdot(\text{column } j \text{ of } B).$$



Place B above and A to the left: each entry of AB sits where a row of A meets a column of B.

$(AB)_{21} = (1)(1) + (-1)(0) + (3)(4) = \mathbf{13}$

Size rule: like dominoes

$$(\mathbf{m \times n})\,(\mathbf{n \times p}) \;\to\; \mathbf{m \times p}$$

must match

The touching ends must be equal; the outer ends give the size.

With A: 2×3, B: 3×4, D: 3×3:

- AB: 2×4; BA: undefined ($4 \neq 2$)
- AD: 2×3; DA: undefined
- A^2: undefined (A^2 needs A square)
- $A^T A$: 3×3 (always defined)

Same two factors, swapped

$$\begin{bmatrix}2 & -1 & 3\end{bmatrix}\begin{bmatrix}1\\4\\2\end{bmatrix} = \begin{bmatrix}4\end{bmatrix}\ (1\times 1), \qquad \begin{bmatrix}1\\4\\2\end{bmatrix}\begin{bmatrix}2 & -1 & 3\end{bmatrix} = \begin{bmatrix}2 & -1 & 3\\8 & -4 & 12\\4 & -2 & 6\end{bmatrix}\ (3\times 3).$$

Row × column gives a number (a dot product); column × row gives a whole matrix.

The column view and linear systems

Key formula

If $A = \begin{bmatrix}\vec{\mathbf{a}}_1 & \vec{\mathbf{a}}_2 & \cdots & \vec{\mathbf{a}}_n\end{bmatrix}$ (its columns), then

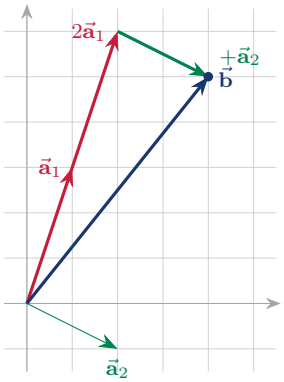
$$A\vec{\mathbf{x}} = x_1\vec{\mathbf{a}}_1 + x_2\vec{\mathbf{a}}_2 + \cdots + x_n\vec{\mathbf{a}}_n.$$

$A\vec{\mathbf{x}}$ is a **combination of the columns** of A.

A system is one matrix equation $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$

$$\begin{cases} x + 2y = 4 \\ 3x - y = 5 \end{cases} \iff \begin{bmatrix}1 & 2\\3 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}4\\5\end{bmatrix}$$

Solving means finding the combination of the columns that reaches $\vec{\mathbf{b}}$. Here $x = 2$, $y = 1$: $2\begin{bmatrix}1\\3\end{bmatrix} + 1\begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}4\\5\end{bmatrix}$ (figure).



Properties of the product

What still works

- $(AB)C = A(BC)$ associativity
- $A(B+C) = AB + AC$, $(A+B)C = AC + BC$
- $k(AB) = (kA)B = A(kB)$
- $I_m A = A = AI_n$, $OA = O$, $AO = O$
- $(AB)^T = B^T A^T$

Distribute **without changing sides**: in $A(B+C)$, A stays on the left.

Expanding correctly

$(A+B)^2 = (A+B)(A+B) = A^2+AB+BA+B^2$. This equals $A^2 + 2AB + B^2$ **only if** $AB = BA$. But $(A+I)^2 = A^2+2A+I$ is *always* true, because $AI = IA$.

The four traps of the product

1. $AB \neq BA$ **in general.**

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}:$$

$$AP = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}, \qquad PA = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}.$$

P on the right swaps the **columns**; on the left, the **rows**.

2. $AB = O$ **with** $A \neq O$ **and** $B \neq O$.

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

3. $AB = AC$ **does not give** $B = C$.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = I_2, \quad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}:$$

$AB = A = AC$, but $B \neq C$. Cancelling A is allowed only when A is **invertible**.

4. $(A+B)^2 \neq A^2 + 2AB + B^2$**.** With A, P of trap 1:

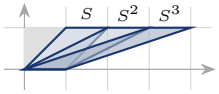
$$AP + PA = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} \neq 2AP = \begin{bmatrix} 4 & 2 \\ 8 & 6 \end{bmatrix}.$$

6 Powers and trace (square matrices)

Powers

For $A \in \mathcal{M}_n$: $A^k = \underbrace{A\,A\cdots A}_{k \text{ factors}}$, $A^0 = I_n$. $A^p A^q = A^{p+q}$ and $(A^p)^q = A^{pq}$, but in general $(AB)^k \neq A^k B^k$.

Shear powers. $S^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$: shearing n times is one shear n times as strong.



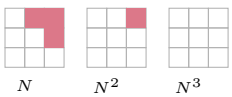
Diagonal matrices are powered entry by entry:

$$\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}^k = \begin{bmatrix} 2^k & 0 \\ 0 & (-1)^k \end{bmatrix}.$$

Only diagonal ones! For other matrices, multiply.

Nilpotent: $N \neq O$ but $N^3 = O$.

$$N = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \quad N^2 = \begin{bmatrix} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



nonzero entries move away from the diagonal until nothing is left

Trace

$\operatorname{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$, the sum of the diagonal.

$$\begin{bmatrix} \mathbf{4} & -1 & 7 \\ 2 & \mathbf{5} & 0 \\ 8 & 3 & \mathbf{-6} \end{bmatrix} \quad \operatorname{tr} = 4 + 5 - 6 = 3$$

Properties. $\operatorname{tr}(A+B) = \operatorname{tr} A + \operatorname{tr} B$, $\operatorname{tr}(kA) = k \operatorname{tr} A$, $\operatorname{tr}(A^T) = \operatorname{tr} A$, $\operatorname{tr}(I_n) = n$, and

$$\operatorname{tr}(AB) = \operatorname{tr}(BA) \quad \textbf{even though } AB \neq BA.$$

Warning

$\operatorname{tr}(AB) \neq \operatorname{tr}(A)\operatorname{tr}(B)$ in general: $\operatorname{tr}(I_2 I_2) = 2$, but $\operatorname{tr}(I_2)\operatorname{tr}(I_2) = 4$.

$\operatorname{tr}(AB) = \operatorname{tr}(BA)$, even with different sizes

With $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 4 & -1 \end{bmatrix}$ from Section 5:

$$AB = \begin{bmatrix} \mathbf{2} & 5 \\ 13 & \mathbf{-2} \end{bmatrix}, \ \operatorname{tr}(AB) = 0, \qquad BA = \begin{bmatrix} \mathbf{4} & -1 & 6 \\ 1 & \mathbf{-1} & 3 \\ 7 & 5 & \mathbf{-3} \end{bmatrix}, \ \operatorname{tr}(BA) = 0.$$

7 Invertible matrices

Inverse

$A \in \mathcal{M}_n$ is **invertible** if there is $B \in \mathcal{M}_n$ with $AB = BA = I_n$. Then B is unique and written A^{-1}.

The 2×2 case

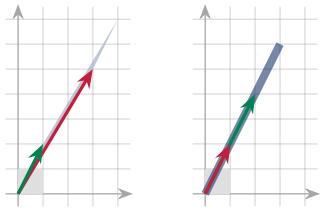
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det A = ad - bc.$$

$$\text{If } ad - bc \neq 0: \quad A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Swap the diagonal, change the sign of the other two, divide. If $ad - bc = 0$, A is **not** invertible.

Never write $\frac{1}{A}$ or $\frac{B}{A}$: $A^{-1}B$ and BA^{-1} are different in general.

Why $\det A = 0$ means no inverse



$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$, $\det = 1$ — still a parallelogram: can be undone

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $\det = 0$ — crushed onto a line: cannot be undone

Parallel columns squash the plane onto a line: many points land on the same image, so there is no way back. (Preview: $|ad - bc|$ is the area of the image of the unit square.)

Solve $3x + y = 5, \quad 5x + 2y = 9$

$A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$, $\det A = 6 - 5 = 1$, $A^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$.

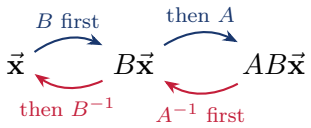
Multiply $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ on the **left** by A^{-1}:

$$\vec{\mathbf{x}} = A^{-1}\vec{\mathbf{b}} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Check: $3 + 2 = 5$ ✓, $5 + 4 = 9$ ✓

Rules (A, B invertible)

$(A^{-1})^{-1} = A$, $(kA)^{-1} = \frac{1}{k}A^{-1}$ $(k \neq 0)$,
$(A^T)^{-1} = (A^{-1})^T$, $(AB)^{-1} = B^{-1}A^{-1}$.



Socks and shoes: what goes on last comes off first. Same reversal for $(AB)^T = B^T A^T$.

8 Cheat sheet

Operation	Defined when	Size	How to compute
$A + B$	same size	$m \times n$	add entry by entry
kA	always	$m \times n$	multiply every entry by k
A^T	always	$n \times m$	$(A^T)_{ij} = a_{ji}$ (rows → columns)
AB	A: $m \times n$, B: $n \times p$	$m \times p$	$(AB)_{ij}$ = row i of A · column j of B
A^k	A square	$n \times n$	$A \cdot A \cdots A$, $A^0 = I_n$
$\operatorname{tr} A$	A square	a number	sum of the diagonal
A^{-1} (2×2)	$ad - bc \neq 0$	2×2	$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Always true

associativity, distributivity (keeping sides), $IA = AI = A$, $\operatorname{tr}(AB) = \operatorname{tr}(BA)$; every rule for $+$ and scalars.

False in general

$AB = BA$; $AB = O \Rightarrow A = O$ or $B = O$; $AB = AC \Rightarrow B = C$; $(A+B)^2 = A^2 + 2AB + B^2$.

9 Self-check

True or false? Justify in one line: a general reason if true, one counterexample if false.

1. If A is 2×3 and B is 3×2, then AB and BA are both defined.
2. Every diagonal matrix is symmetric.
3. If $AB = O$, then $A = O$ or $B = O$.
4. $(AB)^T = A^T B^T$ for all square A, B.
5. $\operatorname{tr}(AB) = \operatorname{tr}(A)\ \operatorname{tr}(B)$.

6. $\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$ is invertible.

7. For every square matrix A, the matrix $A + A^T$ is symmetric.

8. The second column of A is the image of $\vec{\mathbf{j}}$ under A.

Answers

1. **True**: AB is 2×2 and BA is 3×3 (so they can never be equal).
2. **True**: all entries off the diagonal are 0, so $a_{ij} = a_{ji}$.
3. **False**: $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} = O$ with both factors nonzero.
4. **False**: the order reverses, $(AB)^T = B^T A^T$, and $A^T B^T \neq B^T A^T$ in general.
5. **False**: $\operatorname{tr}(I_2 I_2) = 2$, while $\operatorname{tr}(I_2)\operatorname{tr}(I_2) = 4$.
6. **False**: $ad - bc = 4 - 4 = 0$; the columns are parallel, so the plane is crushed onto a line.
7. **True**: $(A + A^T)^T = A^T + (A^T)^T = A^T + A$.
8. **True**: $A\vec{\mathbf{j}} = A \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ picks out column 2.