Chapter 6 — Systems of Linear Equations

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Part A — The essentials

Competence 1 — Linear equations and solutions

Goal. decide whether an equation is linear, and whether a list of numbers is a solution of a system.

Reminder

An equation is **linear** when each unknown appears only to the first power, multiplied by a constant (no products of unknowns, no powers, no unknown inside a function or a denominator). A **solution** of a system satisfies **all** its equations: one failed equation is enough to reject it.

Exercise A1 COMPUTE ⋆

(a) Which equations are linear in x, y, z?

(i) $4x - \sqrt{3}\,y = z$ (ii) $\frac{x}{y} = 2$ (iii) $2^x + y = 0$ (iv) $(x+1)(y-2) = xy$ (v) $x = 3$

(b) For the system $\begin{cases} x + 2y - z = 3 \\ 2x - y + z = 4 \\ x + y + z = 4 \end{cases}$, which of $(2,1,1)$, $(3,-2,-4)$ and $(1,1,0)$ are solutions?

Competence 2 — The augmented matrix and row operations

Goal. translate a system into its augmented matrix (and back), and apply row operations one at a time.

Reminder

Put each equation in the form $ax + by + cz = d$ (unknowns on the left in the same order, constant on the right), and write 0 for a missing unknown. Allowed operations: **swap** $R_i \leftrightarrow R_j$; **scale** $R_i \leftarrow kR_i$ with $k \neq 0$; **replace** $R_i \leftarrow R_i + kR_j$ with $j \neq i$. Do **one** operation at a time, always on the *current* rows.

Exercise A2 COMPUTE ⋆

(a) Write the augmented matrix of $\begin{cases} 2x - z = 4 \\ y + 3 = x \\ z = 2y \end{cases}$.

(b) Write the system (unknowns x_1, x_2, x_3, x_4) whose augmented matrix is $\left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & 5 \\ 0 & 1 & 4 & 0 & -1 \end{array}\right]$.

(c) Starting from $M = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 5 & 1 & 7 \\ 3 & 4 & 1 & 7 \end{array}\right]$, apply in this order $R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 - 3R_1$, $R_3 \leftarrow R_3 + 2R_2$, and write the matrix after each step.

Exercise A3 COMPUTE ⋆

(a) Write the augmented matrix of the system in x_1, x_2, x_3, x_4:

$$\begin{cases} x_1 + 2x_3 = x_4 - 1 \\ 3x_2 = x_1 \\ x_4 = 2 \\ x_2 - x_3 + x_4 = 0 \end{cases}$$

(b) Starting from $N = \left[\begin{array}{ccc|c} 2 & 4 & -2 & 6 \\ 1 & 1 & 1 & 2 \\ 3 & 2 & 4 & 5 \end{array}\right]$, apply in this order $R_1 \leftrightarrow R_2$, $R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 - 3R_1$, $R_2 \leftarrow \frac{1}{2}R_2$, and write the matrix after each step.

(c) Explain why the system of the final matrix in (b) has the same solutions as the system of N.

Competence 3 — Row echelon form and Gaussian elimination

Goal. recognise a row echelon form; reduce an augmented matrix to row echelon form, then solve by back substitution.

Reminder

Row echelon form: zero rows at the bottom; the first nonzero entry of each row (its **pivot**) is strictly to the right of the pivot above.
Method: (1) put a nonzero pivot at the top of the leftmost nonzero column (swap if needed); (2) create zeros below it with replacements; (3) cover that row and repeat; (4) solve from the bottom up. A pivot equal to 1 avoids fractions.

Exercise A4 COMPUTE ⋆ *Recognising a row echelon form*

Which matrices are in row echelon form? For those that are, circle the pivots and count them. For the others, give one row operation that puts them in row echelon form.

$$G_1 = \begin{bmatrix} 2 & -1 & 0 & 3 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad G_2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & 4 \end{bmatrix} \quad G_3 = \begin{bmatrix} 0 & 1 & 5 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$G_4 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad G_5 = \begin{bmatrix} 3 & 1 \\ 0 & 0 \\ 0 & 2 \end{bmatrix} \quad G_6 = \begin{bmatrix} 1 & 4 & -2 & 0 & 1 \end{bmatrix}$$

Exercise A5 COMPUTE ⋆

Solve by Gaussian elimination, then check your solution in the original system:

$$\begin{cases} x + 2y - z = 1 \\ 2x + 3y + z = 7 \\ -x + y + 2z = 4 \end{cases}$$

Exercise A6 COMPUTE ⋆⋆ *Choosing the pivot*

Solve by Gaussian elimination, then check.

$$\text{(a)} \begin{cases} 2y + z = 1 \\ x + y - z = 3 \\ 2x + 4y + z = 5 \end{cases} \qquad \text{(b)} \begin{cases} 3x - y + 2z = 7 \\ x + 2y - z = -1 \\ 2x + y + 3z = 8 \end{cases}$$

In (a), the entry in position $(1, 1)$ is 0: what must you do first? In (b), choose your first pivot so as to avoid fractions.

Exercise A7 COMPUTE ⋆⋆ *Four unknowns*

Solve by Gaussian elimination, then check:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 2 \\ x_1 + 2x_2 + x_3 + 2x_4 = 1 \\ 2x_1 + x_2 + 3x_3 + x_4 = 7 \\ x_1 + x_2 + 2x_3 + 3x_4 = 4 \end{cases}$$

Competence 4 — Reduced row echelon form

Goal. recognise the RREF and reach it with Gauss–Jordan.

Reminder

RREF: a row echelon form in which every pivot is 1 and is the **only** nonzero entry of its column. **Gauss–Jordan:** after going down, go back up: make each pivot 1, then clear the entries above it, starting from the last pivot. A matrix has many echelon forms, but only **one** RREF.

Exercise A8 COMPUTE ⋆

(a) Which of these matrices are in RREF? For the others, give one row operation that brings them closer to RREF.

$$P = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix} \quad S = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

(b) Start from the echelon form you obtained in Exercise A5 and continue with Gauss–Jordan until the RREF. Read the solution directly.

Exercise A9 COMPUTE ⋆⋆ *Gauss–Jordan from start to finish*

(a) Solve with Gauss–Jordan (go down, then go up until the RREF): $\begin{cases} x+y+z=2 \\ 2x-y+z=6 \\ x+2y-z=-1 \end{cases}$.

(b) Reduce to RREF: $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & 0 \\ 3 & 6 & 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 5 \\ 1 & 0 & 3 \end{bmatrix}$.

Competence 5 — How many solutions? Free variables

Goal. decide between no solution, one solution and infinitely many; find the free variables; write the solution set in parametric form and describe it geometrically.

Reminder

In the echelon form of $[\,A \mid \vec{\mathbf{b}}\,]$:

1. a row $[\,0 \ \cdots \ 0 \mid c\,]$ with $c \neq 0$ means $0 = c$: **no solution**;
2. otherwise, look at the columns of A only: a column **without** a pivot gives a **free variable**. No free variable: **one solution**. At least one: **infinitely many**.

Give each free variable a parameter (s, t, …) and express the pivot variables with it. Number of parameters = (number of unknowns) − (number of pivots).

Warning

A zero row $[\,0 \ \cdots \ 0 \mid 0\,]$ does **not** mean "infinitely many solutions". Only the free variables decide.

Exercise A10 COMPUTE ⋆ *Reading the RREF*

Each matrix is the RREF of the augmented matrix of a system in x, y, z (in (f): x_1, x_2, x_3, x_4). Give the number of solutions, the solution set (in parametric form when needed), and describe it geometrically (point, line, plane, empty).

(a) $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$ (b) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ (c) $\left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$

(d) $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ (e) $\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \end{array}\right]$ (f) $\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$

Exercise A11 COMPUTE ⋆⋆ *Same coefficients, two right-hand sides*

Consider the planes $\mathcal{P}_1 : x - y + 2z = 3$, $\mathcal{P}_2 : 2x + y + z = 6$ and $\mathcal{P}_3 : 3x + 3z = c$.

(a) Solve the system $\{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3\}$ for $c = 9$. Write the solution set in parametric form and check two of its points.

(b) Check that its direction vector is perpendicular to the three normal vectors, and compare it with $(1,-1,2)\times(2,1,1)$.

(c) Solve the system for $c=10$. Describe the position of the three planes.

Competence 6 — Rank and dimension of the solution set

Goal. compute the rank of a matrix; decide from the ranks whether a system has solutions, how many free variables it has, and the dimension of its solution set.

Reminder

The **rank** of a matrix M, written $\operatorname{rank} M$, is the **number of pivots** in an echelon form of M (all echelon forms of M have the same number of pivots). For a system $A\vec{\mathbf{x}}=\vec{\mathbf{b}}$ with n unknowns, reduce $[\,A \mid \vec{\mathbf{b}}\,]$ once and count: $\operatorname{rank} A$ (pivots among the columns of A) and $\operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,]$ (pivots in all the columns).

1. The system is consistent $\iff \operatorname{rank} A = \operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,]$.
2. It then has $n-\operatorname{rank} A$ free variables: its solution set has **dimension** $n-\operatorname{rank} A$ (0: a point, 1: a line, 2: a plane).

Since there is at most one pivot per row and per column, $\operatorname{rank} A \le$ (number of rows) and $\operatorname{rank} A \le$ (number of columns).

Exercise A12 COMPUTE ⋆ *Computing ranks*

Give the rank of each matrix.

(a) $\begin{bmatrix} 2 & -4 \\ -3 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & 3 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} 0 & 3 & -1 & 2 \end{bmatrix}$

Exercise A13 COMPUTE ⋆⋆ *From the ranks to the solution set*

For each system: compute $\operatorname{rank} A$ and $\operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,]$, decide whether it is consistent, give the number of free variables and the dimension of the solution set. Then write the solution set when it is not empty.

(a) $\begin{cases} x+y-z=1 \\ 2x+2y-2z=2 \\ -x-y+z=-1 \end{cases}$ (b) $\begin{cases} x+2y=3 \\ 2x+3y=5 \\ 4x+7y=11 \end{cases}$ (c) $\begin{cases} x-y+z=2 \\ 2x+y-z=1 \\ 4x-y+z=5 \end{cases}$

(d) $\begin{cases} x+y+z=1 \\ x-y+z=3 \\ 2x+2z=5 \end{cases}$ (e) $\begin{cases} x_1+x_2-x_3+x_4=2 \\ x_2+x_3-x_4=1 \end{cases}$

Exercise A14 THINK ⋆⋆ *Reasoning with the ranks only*

No computation is needed: use only the facts of the reminder.

(a) A system has 4 equations, 3 unknowns, $\operatorname{rank} A = 3$ and $\operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,] = 3$. Describe its solution set.

(b) A system has 3 equations, 4 unknowns and $\operatorname{rank} A = 3$. Explain why it is consistent for every $\vec{\mathbf{b}}$, and give the dimension of its solution set.

(c) A system has 3 equations, 3 unknowns, $\operatorname{rank} A = 1$ and $\operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,] = 1$. Describe its solution set geometrically.

(d) A system has 2 equations, 3 unknowns, $\operatorname{rank} A = 1$ and $\operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,] = 2$. Describe the two planes.

(e) Can a system of 3 equations in 5 unknowns have exactly one solution?

Competence 7 — Homogeneous systems

Goal. solve $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$, and use "particular + homogeneous" to describe all the solutions of $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

Reminder

$A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ always has the trivial solution $\vec{\mathbf{0}}$; it has other solutions exactly when there is a free variable. If $\vec{\mathbf{p}}$ is **one** solution of $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$, all the solutions are $\vec{\mathbf{x}} = \vec{\mathbf{p}} + \vec{\mathbf{h}}$ with $A\vec{\mathbf{h}} = \vec{\mathbf{0}}$.

Exercise A15 COMPUTE ⋆⋆

(a) Solve the homogeneous system $\begin{cases} x - y + 2z = 0 \\ 2x - y + 5z = 0 \end{cases}$ and describe its solution set geometrically.

(b) Check that $\vec{\mathbf{p}} = (3, 2, 0)$ is a solution of $\begin{cases} x - y + 2z = 1 \\ 2x - y + 5z = 4 \end{cases}$. Using (a), write *all* its solutions without any new elimination.

(c) Is $(0, 1, 1)$ a solution of the system in (b)? Answer in two ways: by substitution, and with the formula of (b).

Competence 8 — Systems with a parameter

Goal. find the values of a parameter for which a system has no solution, one solution or infinitely many.

Reminder

Eliminate with pivots that do **not** contain the parameter (swap if needed), factor what ends up in pivot position, then split into cases. Never divide by an expression that may be 0.

Exercise A16 COMPUTE ⋆⋆

Discuss, according to k, the number of solutions of $\begin{cases} kx + 4y = 2 \\ x + ky = 1 \end{cases}$.

(a) Give the solution when it is unique, and check it for $k = 0$.

(b) For each special value of k, describe the two lines.

Competence 9 — The inverse with Gauss–Jordan

Goal. compute A^{-1} with $[\,A \mid I\,]$, or show that A is not invertible; decide invertibility with the rank.

Reminder

$[\,A \mid I\,] \xrightarrow{\text{Gauss–Jordan}} [\,I \mid A^{-1}\,]$. If a row of zeros appears on the left, A is **not** invertible. Check one or two entries of $AA^{-1} = I$.
With the rank: a square matrix A of order n is invertible $\iff$ $\operatorname{rank} A = n$ (a pivot in every row and every column, so the RREF is I_n).

Exercise A17 COMPUTE ⋆⋆

Find the inverse, or show that it does not exist. In (a), compare with the 2×2 formula.

$$\text{(a) } A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \qquad \text{(b) } B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \qquad \text{(c) } C = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix}$$

Exercise A18 COMPUTE ⋆⋆ *Invertible or not? Use the rank*

Without computing any inverse, give the rank of each matrix and decide whether it is invertible.

$$\text{(a) } \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} \quad \text{(b) } \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad \text{(c) } \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{(d) } \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{(e) } \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \\ 2 & 1 & -2 \end{bmatrix}$$

(f) For which values of k is $\begin{bmatrix} 1 & k \\ k & 4 \end{bmatrix}$ invertible? Answer with the rank.

Competence 10 — Modelling

Goal. translate a situation into a linear system, solve it, and interpret the answer.

Reminder

Name the unknowns (with units), translate each condition into **one** equation, solve, then check that the answer makes sense in the situation.

Exercise A19 COMPUTE ⋆ *Theatre tickets*

A theatre sold 500 tickets: adult (\$15), student (\$10) and child (\$6), for a total of \$5 520. There were 100 fewer adult tickets than student and child tickets together. How many tickets of each kind were sold?

Part B — Going further

Exercise B1 TRUE / FALSE ⋆⋆ *Solutions and row operations*

If a statement is true, give a short general argument; if it is false, give one explicit counterexample.

(a) A system of two linear equations in two unknowns can have exactly two solutions.

(b) Adding an equation to a system can never create a new solution.

(c) The operation $R_2 \leftarrow 3R_2 - R_1$ does not change the solution set.

(d) If a row $[\,0\ 0\ 0 \mid 0\,]$ appears during elimination, an equation has been lost and the solution set has changed.

(e) Three planes whose normal vectors are pairwise non-parallel always have exactly one common point.

Exercise B2 THINK ⋆⋆ *Spot the error*

To solve $\begin{cases} x + y = 2 \\ x - y = 0 \end{cases}$, a student writes in a single step "$R_1 \leftarrow R_1 - R_2$ and $R_2 \leftarrow R_2 - R_1$" and obtains $\left[\begin{array}{cc|c} 0 & 2 & 2 \\ 0 & -2 & -2 \end{array}\right]$. He concludes that $y = 1$ and that x can be anything.

(a) Solve the system correctly.

(b) Check that $(5, 1)$ satisfies the student's system but not the original one. What went wrong?

Exercise B3 THINK ⋆⋆ *Two students, two answers?*

Two students reduce $M = \left[\begin{array}{cc|c} 3 & 6 & 9 \\ 1 & 4 & 7 \end{array}\right]$ to echelon form:

$$\text{Student 1: } \left[\begin{array}{cc|c} 3 & 6 & 9 \\ 0 & 2 & 4 \end{array}\right] \qquad \text{Student 2: } \left[\begin{array}{cc|c} 1 & 4 & 7 \\ 0 & -6 & -12 \end{array}\right]$$

(a) Find the operations each student used. Are both matrices in row echelon form?

(b) Reduce both matrices to RREF. What do you notice, and why is it not a coincidence?

Exercise B4 THINK ⋆⋆ *The intersection of two planes*

Let $\mathcal{P}_1 : x + 2y - z = 3$ and $\mathcal{P}_2 : 2x + 3y + z = 4$.

(a) Find the line $\mathcal{P}_1 \cap \mathcal{P}_2$ by elimination. Check your direction vector with a cross product.

(b) Find the point where this line meets the plane $x + y + z = 0$.

(c) Write the equation of a third plane, different from $\mathcal{P}_1$ and $\mathcal{P}_2$, that contains the whole line.

Exercise B5 THINK ⋆⋆⋆ *Build the system*

Give an example of each, and justify.

(a) A system of two equations whose solution set is the line $(1, 0, 2) + t(1, 1, 0)$.

(b) A system of three equations in x, y, z, with no two planes parallel, and no solution.

(c) A system of two equations in three unknowns with no solution.

Exercise B6 COMPUTE ⋆⋆ *Two parameters*

Discuss, according to a and b, the number of solutions of $\begin{cases} x + y + 2z = 1 \\ 2x + 3y + 5z = 4 \\ x + 2y + az = b \end{cases}$. Give the solution set in parametric form in the case with infinitely many solutions, and check it.

Exercise B7 THINK ⋆⋆ *Spot the error*

A student discusses $\begin{cases} x + y = 2 \\ x + ky = k + 1 \end{cases}$:

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & k & k+1 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & k-1 & k-1 \end{array}\right] \xrightarrow{R_2 \leftarrow \frac{1}{k-1} R_2} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

and concludes: "for every k, the unique solution is $(1, 1)$". Find the error and give the correct discussion.

Exercise B8 COMPUTE ⋆⋆ *An invertible matrix?*

For which values of k is $E = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & k \end{bmatrix}$ invertible? For the value(s) where it is not, find a nonzero solution of $E\vec{\mathbf{x}} = \vec{\mathbf{0}}$.

Exercise B9 PROVE ⋆⋆ *Two solutions are never alone*

Let $\vec{\mathbf{x}}_1 \neq \vec{\mathbf{x}}_2$ be two solutions of $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

(a) Show that $\vec{\mathbf{x}}_1 + t(\vec{\mathbf{x}}_2 - \vec{\mathbf{x}}_1)$ is a solution for every real t, and deduce that a linear system cannot have exactly two solutions.

(b) Show that $\vec{\mathbf{x}}_1 - \vec{\mathbf{x}}_2$ is a solution of $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$.

(c) Show that $\vec{\mathbf{x}}_1 + \vec{\mathbf{x}}_2$ is a solution of $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ only when $\vec{\mathbf{b}} = \vec{\mathbf{0}}$.

Exercise B10 COMPUTE ⋆⋆ *A circle through three points*

A circle has an equation of the form $x^2 + y^2 + ax + by + c = 0$.

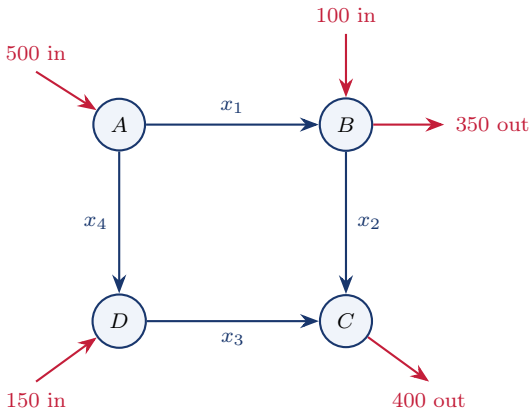
(a) Find the circle through $A(1,1)$, $B(3,1)$ and $C(2,2)$. Give its centre and its radius by completing the squares.

(b) Why is the system linear, although a circle is not a line?

(c) Show that no circle passes through $(0,0)$, $(1,1)$ and $(2,2)$. Interpret.

Exercise B11 COMPUTE ⋆⋆ *Traffic in a city centre*

The figure shows four one-way streets around a block, with the hourly traffic entering and leaving the network. At each intersection, the number of cars entering equals the number leaving.



(a) Write one equation per intersection, and solve the system.

(b) Explain the meaning of the free variable in terms of the cars' routes.

(c) Flows cannot be negative. Give the possible values of each flow.

(d) Street BC can carry at most 200 cars per hour. What does this impose on the traffic in street AD?

Exercise B12 THINK ⋆⋆⋆ *Synthesis: two solutions, many consequences*

A is a 3×3 matrix, and the system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has the two solutions $(1,2,0)$ and $(3,1,1)$.

(a) Give infinitely many other solutions of $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

(b) Compute $A\,(2,-1,1)$.

(c) Is A invertible?

(d) Can $A\vec{\mathbf{x}} = \vec{\mathbf{c}}$ have exactly one solution for some other vector $\vec{\mathbf{c}}$?