Chapter 6 - Detailed solutions

Systems of linear equations: elimination, geometry, inverses and applications

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Part A — The essentials

Solution A1 COMPUTE ⋆

Method. An equation is linear when each unknown appears only to the first power, multiplied by a constant. Simplify first if needed. For a solution, substitute in *every* equation; one failed equation is enough to reject.

(a)
- (i) $4x - \sqrt{3}\,y = z$, i.e. $4x - \sqrt{3}\,y - z = 0$: **linear**. $\sqrt{3}$ is just a number.
- (ii) $\frac{x}{y} = 2$: **not linear**, y is in a denominator.
- (iii) $2^x + y = 0$: **not linear**, the unknown x is an exponent.
- (iv) $(x+1)(y-2) = xy$ looks quadratic, so expand: $xy - 2x + y - 2 = xy \iff -2x + y = 2$. The product xy cancels: **linear**.
- (v) $x = 3$, i.e. $1x + 0y + 0z = 3$: **linear**.

(b)

	$x + 2y - z = 3$	$2x - y + z = 4$	$x + y + z = 4$	solution?
$(2, 1, 1)$	$2 + 2 - 1 = 3$ ✓	$4 - 1 + 1 = 4$ ✓	$2 + 1 + 1 = 4$ ✓	**yes**
$(3, -2, -4)$	$3 - 4 + 4 = 3$ ✓	$6 + 2 - 4 = 4$ ✓	$3 - 2 - 4 = -3$ ×	**no**
$(1, 1, 0)$	$1 + 2 - 0 = 3$ ✓	$2 - 1 + 0 = 1$ ×	(not needed)	**no**

$(3, -2, -4)$ is on the first two planes (so on their line of intersection) but not on the third one.

Common mistake

Deciding from the *look* of an equation. Always simplify first: (iv) contains a product, yet it is linear.

Solution A2 COMPUTE ⋆

Method. Put every equation in the form $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = b$: unknowns on the left in order, constant on the right; 0 for a missing unknown. Apply the operations one at a time, always to the *current* matrix.

(a) Rewrite each equation in the form $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = b$ (unknowns on the left, in order; constant on the right):

$$\begin{array}{lcl} x_1 + 2x_3 = x_4 - 1 & \longrightarrow & x_1 + 0x_2 + 2x_3 - x_4 = -1 \\ 3x_2 = x_1 & \longrightarrow & -x_1 + 3x_2 + 0x_3 + 0x_4 = 0 \\ x_4 = 2 & \longrightarrow & 0x_1 + 0x_2 + 0x_3 + x_4 = 2 \\ x_2 - x_3 + x_4 = 0 & \longrightarrow & 0x_1 + x_2 - x_3 + x_4 = 0 \end{array} \qquad \left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & -1 \\ -1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right].$$

(b) Two rows give two equations; four coefficient columns give four unknowns:

$$\begin{cases} x_1 \qquad - 2x_3 + 3x_4 = 5 \\ \qquad x_2 + 4x_3 \qquad\quad = -1 \end{cases}$$

(c) One operation at a time, computing each new row on the side:

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 6 \\ 1 & 1 & 1 & 2 \\ 3 & 2 & 4 & 5 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 4 & -2 & 6 \\ 3 & 2 & 4 & 5 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 2 & -4 & 2 \\ 3 & 2 & 4 & 5 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - 3R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 2 & -4 & 2 \\ 0 & -1 & 1 & -1 \end{array}\right] \xrightarrow{R_2 \leftarrow \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 1 & -1 \end{array}\right].$$

$((2,4,-2,6) - 2(1,1,1,2) = (0,2,-4,2);\ \ (3,2,4,5) - 3(1,1,1,2) = (0,-1,1,-1).)$

(d) Each operation is elementary and can be undone ($R_1 \leftrightarrow R_2$ by itself, $R_i \leftarrow R_i - kR_1$ by $R_i \leftarrow R_i + kR_1$, $R_2 \leftarrow \frac{1}{2}R_2$ by $R_2 \leftarrow 2R_2$). So the original system can be recovered from the new one: no information is lost and the two systems have the same solutions.

Common mistake

Moving a term without changing its sign. In $x_1 + 2x_3 = x_4 - 1$, the x_4 goes to the left as $-x_4$; in $3x_2 = x_1$, the x_1 goes to the left as $-x_1$ and the constant is 0.

Takeaway. The swap was chosen to get a pivot equal to 1 in position $(1,1)$: with it, the replacements use whole numbers only.

Solution A3 COMPUTE ⋆ *Recognising a row echelon form*

Method. Check the two conditions: zero rows at the bottom, and each pivot strictly to the right of the pivot of the row above.

$$G_1 = \begin{bmatrix} \mathbf{2} & -1 & 0 & 3 \\ 0 & 0 & \mathbf{4} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad G_3 = \begin{bmatrix} 0 & \mathbf{1} & 5 \\ 0 & 0 & \mathbf{2} \\ 0 & 0 & 0 \end{bmatrix} \qquad G_6 = \begin{bmatrix} \mathbf{1} & 4 & -2 & 0 & 1 \end{bmatrix}$$

- G_1: REF, **2 pivots**. The step from row 1 to row 2 skips column 2: allowed.
- G_3: REF, **2 pivots**. A first column of zeros is allowed.
- G_2: not in REF, the pivot of row 3 (column 2) is to the *left* of the pivot of row 2 (column 3). Fix: $R_2 \leftrightarrow R_3$.
- G_4: not in REF, the entry 2 is below the first pivot. Fix: $R_2 \leftarrow R_2 - 2R_1$, which gives I_3.
- G_5: not in REF, the zero row is not at the bottom. Fix: $R_2 \leftrightarrow R_3$.

Common mistake

Thinking that the pivots of a REF must be 1, or that the pivots must be on the diagonal. Neither is required: only the staircase shape matters.

Solution A4 COMPUTE ⋆

Method. The entry in position $(1,1)$ is 1: it is the first pivot, no swap needed. Zeros below it, then below the second pivot, then back substitution.

$$\left[\begin{array}{ccc|c} \mathbf{1} & 2 & -1 & 1 \\ 2 & 3 & 1 & 7 \\ -1 & 1 & 2 & 4 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 + R_1]{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & \mathbf{-1} & 3 & 5 \\ 0 & 3 & 1 & 5 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 + 3R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & 5 \\ 0 & 0 & \mathbf{10} & 20 \end{array}\right]$$

On the side: $(2,3,1,7) - 2(1,2,-1,1) = (0,-1,3,5)$, $(-1,1,2,4) + (1,2,-1,1) = (0,3,1,5)$, $(0,3,1,5) + 3(0,-1,3,5) = (0,0,10,20)$.

Back substitution. $10z = 20 \Rightarrow z = 2$; $-y + 3z = 5 \Rightarrow y = 6 - 5 = 1$; $x + 2y - z = 1 \Rightarrow x = 1 - 2 + 2 = 1$.

$$(x, y, z) = (1, 1, 2).$$

Check. $1+2-2 = 1$ ✓, $2+3+2 = 7$ ✓, $-1+1+4 = 4$ ✓. Three pivots for three unknowns: the three planes meet at exactly one point.

Solution A5 COMPUTE ⋆⋆ *Choosing the pivot*

Method. Choose the pivot first (swap if needed), create zeros below it, cover the pivot row, repeat; then back substitution.

(a) The entry in position $(1,1)$ is 0: it cannot be a pivot. Swap with row 2, whose first entry is 1.

$$\left[\begin{array}{ccc|c} 0 & 2 & 1 & 1 \\ 1 & 1 & -1 & 3 \\ 2 & 4 & 1 & 5 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} \mathbf{1} & 1 & -1 & 3 \\ 0 & 2 & 1 & 1 \\ 2 & 4 & 1 & 5 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & \mathbf{2} & 1 & 1 \\ 0 & 2 & 3 & -1 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & \mathbf{2} & -2 \end{array}\right]$$

$2z = -2 \Rightarrow z = -1$; $2y + z = 1 \Rightarrow y = 1$; $x + y - z = 3 \Rightarrow x = 3 - 1 - 1 = 1$. Solution: $(1, 1, -1)$.

Check. $2(1) + (-1) = 1$ ✓, $1 + 1 + 1 = 3$ ✓, $2 + 4 - 1 = 5$ ✓.

(b) Row 2 starts with 1: swap it to the top, so that the first pivot is 1 and the multipliers are whole numbers.

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 7 \\ 1 & 2 & -1 & -1 \\ 2 & 1 & 3 & 8 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} \mathbf{1} & 2 & -1 & -1 \\ 3 & -1 & 2 & 7 \\ 2 & 1 & 3 & 8 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & -7 & 5 & 10 \\ 0 & -3 & 5 & 10 \end{array}\right]$$

Keeping 3 as the first pivot would have required $R_2 \leftarrow R_2 - \frac{1}{3}R_1$ and $R_3 \leftarrow R_3 - \frac{2}{3}R_1$: fractions from the first step. (Finishing the elimination gives $(x, y, z) = (1, 0, 2)$.)

Geometry. In (a), the three planes meet at **exactly one point**: three pivots for three unknowns, no free variable.

Takeaway. Any sequence of valid operations is correct. Good choices (pivots equal to ± 1, small multipliers) only make the arithmetic easier and the mistakes rarer.

Solution A6 COMPUTE ⋆⋆ *Four unknowns*

Method. Same algorithm as for three unknowns: one column at a time, zeros below each pivot.

$$\left[\begin{array}{cccc|c} \mathbf{1} & 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 3 & 1 & 7 \\ 1 & 1 & 2 & 3 & 4 \end{array}\right] \xrightarrow[R_4 \leftarrow R_4 - R_1]{R_2 \leftarrow R_2 - R_1,\ R_3 \leftarrow R_3 - 2R_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & \mathbf{1} & 0 & 1 & -1 \\ 0 & -1 & 1 & -1 & 3 \\ 0 & 0 & 1 & 2 & 2 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 + R_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & \mathbf{1} & 0 & 2 \\ 0 & 0 & 1 & 2 & 2 \end{array}\right] \xrightarrow{R_4 \leftarrow R_4 - R_3} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & \mathbf{2} & 0 \end{array}\right].$$

Back substitution. $2x_4 = 0 \Rightarrow x_4 = 0$; $x_3 = 2$; $x_2 + x_4 = -1 \Rightarrow x_2 = -1$; $x_1 = 2 - x_2 - x_3 - x_4 = 2 + 1 - 2 - 0 = 1$.

$$(x_1, x_2, x_3, x_4) = (1, -1, 2, 0).$$

Check. $1 - 1 + 2 + 0 = 2$ ✓, $1 - 2 + 2 + 0 = 1$ ✓, $2 - 1 + 6 + 0 = 7$ ✓, $1 - 1 + 4 + 0 = 4$ ✓.

Solution A7 COMPUTE ⋆⋆

(a)
- P is in RREF: pivots 1, alone in their columns.
- Q is in RREF: pivots in columns 1 and 3. The 3 is in column 2, which has **no** pivot: it is allowed.
- R is not: the second pivot is 2. $R_2 \leftarrow \frac{1}{2}R_2$ gives $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & \frac{1}{2} \end{bmatrix}$, which is in RREF.
- S is not: the 5 sits above the second pivot. $R_1 \leftarrow R_1 - 5R_2$ gives $\begin{bmatrix} 1 & 0 & -15 \\ 0 & 1 & 3 \end{bmatrix}$.

(b) Start from the echelon form of Exercise A4. Make the pivots 1, then clear above them, from the last pivot up:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & 5 \\ 0 & 0 & 10 & 20 \end{array}\right] \xrightarrow[R_2 \leftarrow -R_2]{R_3 \leftarrow \frac{1}{10}R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & \mathbf{1} & 2 \end{array}\right] \xrightarrow[R_1 \leftarrow R_1 + R_3]{R_2 \leftarrow R_2 + 3R_3} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & \mathbf{1} & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 - 2R_2} \left[\begin{array}{ccc|c} \mathbf{1} & 0 & 0 & 1 \\ 0 & \mathbf{1} & 0 & 1 \\ 0 & 0 & \mathbf{1} & 2 \end{array}\right] \iff x = 1,\ y = 1,\ z = 2.$$

The same solution as in A4, read directly.

(c)
$$\begin{bmatrix}\mathbf{1}&2&-1&3\\2&4&1&0\\3&6&0&3\end{bmatrix}\xrightarrow[R_3\leftarrow R_3-3R_1]{R_2\leftarrow R_2-2R_1}\begin{bmatrix}1&2&-1&3\\0&0&3&-6\\0&0&3&-6\end{bmatrix}\xrightarrow{R_3\leftarrow R_3-R_2}\begin{bmatrix}1&2&-1&3\\0&0&\mathbf{3}&-6\\0&0&0&0\end{bmatrix}$$
$$\xrightarrow{R_2\leftarrow\frac{1}{3}R_2}\begin{bmatrix}1&2&-1&3\\0&0&1&-2\\0&0&0&0\end{bmatrix}\xrightarrow{R_1\leftarrow R_1+R_2}\begin{bmatrix}\mathbf{1}&2&0&1\\0&0&\mathbf{1}&-2\\0&0&0&0\end{bmatrix}.$$

The second pivot is in column 3: column 2 has no pivot, so the 2 in it stays.

$$\begin{bmatrix}\mathbf{1}&-1&2\\2&-1&5\\1&0&3\end{bmatrix}\xrightarrow[R_3\leftarrow R_3-R_1]{R_2\leftarrow R_2-2R_1}\begin{bmatrix}1&-1&2\\0&\mathbf{1}&1\\0&1&1\end{bmatrix}\xrightarrow{R_3\leftarrow R_3-R_2}\begin{bmatrix}1&-1&2\\0&1&1\\0&0&0\end{bmatrix}\xrightarrow{R_1\leftarrow R_1+R_2}\begin{bmatrix}\mathbf{1}&0&3\\0&\mathbf{1}&1\\0&0&0\end{bmatrix}.$$

Takeaway. Going up is back substitution written on the matrix. Classmates may find different echelon forms, but the RREF, like the solution, is the same for everybody.

Solution A8 COMPUTE ⋆ *Reading the RREF*

Method. (1) Is there a row $[\,0\cdots0 \mid c\neq 0\,]$? Then no solution. (2) Otherwise, the columns without pivot give the free variables: give each one a parameter. (3) Solve each pivot row for its pivot variable.

(a) Pivots in columns 1 and 2; $z=t$ free. Rows: $x+3z=2$, $y-z=4$.

$$(x,y,z)=(2-3t,\ 4+t,\ t)=(2,4,0)+t(-3,1,1):\quad \text{infinitely many, a } \textbf{line}.$$

(b) Three pivots, no contradiction: **one solution** $(5,-2,1)$, a point. The zero row only says that one of the four equations was redundant.

(c) Pivots in columns 1 and 3; the *middle* variable $y=t$ is free. Rows: $x-2y=1$, $z=3$.

$$(x,y,z)=(1+2t,\ t,\ 3)=(1,0,3)+t(2,1,0):\quad \text{a } \textbf{line}.$$

(d) The last row reads $0x+0y+0z=1$, i.e. $0=1$: **no solution**, the set is empty.

(e) One pivot (column 1); $y=s$, $z=t$ free: $x=5-3s+2t$.

$$(x,y,z)=(5,0,0)+s(-3,1,0)+t(2,0,1):\quad \text{a } \textbf{plane}.$$

(f) Pivots in columns 1, 2 and 4; $x_3=t$ free. Rows: $x_1+2x_3=3$, $x_2-x_3=1$, $x_4=-2$.

$$(x_1,x_2,x_3,x_4)=(3,1,0,-2)+t(-2,1,1,0).$$

Check. Number of parameters = unknowns − pivots: (a) $3-2=1$, (b) $3-3=0$, (c) $3-2=1$, (e) $3-1=2$, (f) $4-3=1$ ✓.

Common mistakes

Concluding "infinitely many" because of the zero row in (b). Taking the *last* variable as free in (c): the free variables are the columns *without pivot*, wherever they are.

Solution A9 COMPUTE ⋆⋆ *Same coefficients, two right-hand sides*

Method. Eliminate once with the letter c in the last column: the same operations serve both values of c.

$$\left[\begin{array}{ccc|c} \mathbf{1} & -1 & 2 & 3 \\ 2 & 1 & 1 & 6 \\ 3 & 0 & 3 & c \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - 3R_1]{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & \mathbf{3} & -3 & 0 \\ 0 & 3 & -3 & c-9 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & c-9 \end{array}\right]$$

(a) $c = 9$: the last row is $[\,0\ 0\ 0 \mid 0\,]$, no contradiction. Continue to the RREF:

$$\xrightarrow{R_2 \leftarrow \frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 + R_2} \left[\begin{array}{ccc|c} \mathbf{1} & 0 & 1 & 3 \\ 0 & \mathbf{1} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

$z = t$ is free: $x = 3 - t$, $y = t$:

$$(x, y, z) = (3, 0, 0) + t(-1, 1, 1), \qquad t \in \mathbb{R}: \quad \text{a line.}$$

Check. $t = 0$, $(3, 0, 0)$: $3 = 3$, $6 = 6$, $9 = 9$ ✓. $t = 1$, $(2, 1, 1)$: $2-1+2 = 3$, $4+1+1 = 6$, $6 + 3 = 9$ ✓.

(b) $c = 10$: the last row is $[\,0\ 0\ 0 \mid 1\,]$, i.e. $0 = 1$: **no solution**. No two normals are parallel, so the planes meet two by two along lines; these three lines all have direction $(-1, 1, 1)$ and have no common point: a **prism**.



$c = 9$: a common line $c = 10$: a prism

(schematic pictures)

Takeaway. $\mathcal{P}_3$ is the sum of $\mathcal{P}_1$ and $\mathcal{P}_2$ on the left. Whether the constants follow the same sum $(3 + 6 = 9)$ decides between "book" and "prism". Changing only a constant moves a plane parallel to itself.

Solution A10 COMPUTE ⋆ *Computing ranks*

Method. Reduce to an echelon form (not necessarily reduced) and count the pivots.

(a) $R_2 \leftarrow R_2 + \frac{3}{2}R_1$ gives $\begin{bmatrix} \mathbf{2} & -4 \\ 0 & 0 \end{bmatrix}$: **rank 1**. (Row 2 is $-\frac{3}{2}$ times row 1.)

(b) $R_2 - 2R_1$, $R_3 - 3R_1$ give $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, then $R_3 - R_2$: $\begin{bmatrix} \mathbf{1} & 2 & 1 \\ 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 \end{bmatrix}$: **rank 2**.

(c) $R_3 - R_1$, then $R_3 - R_2$: $\begin{bmatrix} \mathbf{1} & 0 & 2 & 1 \\ 0 & \mathbf{1} & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$: **rank 2**. (Row 3 is row 1 + row 2.)

(d) $R_2 - R_1$, $R_3 - R_1$: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{bmatrix}$, then $R_3 - 2R_2$: $\begin{bmatrix} \mathbf{1} & 1 & 1 \\ 0 & \mathbf{1} & 3 \\ 0 & 0 & \mathbf{2} \end{bmatrix}$: **rank 3**.

(e) The zero matrix has no pivot: **rank 0**.

Takeaway. The rank is at most the number of rows and at most the number of columns: here (c) is 3×4, so its rank is at most 3.

Solution A11 COMPUTE ⋆⋆ *From the ranks to the solution set*

Method. Reduce $[\,A \mid \vec{\mathbf{b}}\,]$ once. Count the pivots left of the bar ($\operatorname{rank} A$) and in the whole matrix ($\operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,]$). If they are equal, the dimension is $n - \operatorname{rank} A$.

(a) $R_2 - 2R_1$, $R_3 + R_1$: $\left[\begin{array}{ccc|c} \mathbf{1} & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. $\operatorname{rank} A = 1 = \operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,]$: consistent; $3 - 1 = 2$ free variables ($y = s$, $z = t$); dimension 2:

$$(x, y, z) = (1, 0, 0) + s(-1, 1, 0) + t(1, 0, 1) : \quad \text{a plane (three times the same plane).}$$

(b) $R_2 - 2R_1$, $R_3 - 4R_1$: $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{array}\right]$, then $R_3 - R_2$: $\left[\begin{array}{cc|c} \mathbf{1} & 2 & 3 \\ 0 & \mathbf{-1} & -1 \\ 0 & 0 & 0 \end{array}\right]$. $\operatorname{rank} A = 2 = \operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,]$, $n = 2$: no free variable, dimension 0: the point $(1, 1)$. Three lines through one point: more equations than unknowns does not prevent a solution.

(c) $R_2 - 2R_1$, $R_3 - 4R_1$: $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 3 & -3 & -3 \\ 0 & 3 & -3 & -3 \end{array}\right]$, then $R_3 - R_2$ and $R_2 \leftarrow \frac{1}{3}R_2$, $R_1 \leftarrow R_1 + R_2$: $\left[\begin{array}{ccc|c} \mathbf{1} & 0 & 0 & 1 \\ 0 & \mathbf{1} & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$. $\operatorname{rank} A = 2 = \operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,]$; $3 - 2 = 1$ free variable ($z = t$); dimension 1:

$$(x, y, z) = (1, -1, 0) + t(0, 1, 1) : \quad \text{a line.}$$

(d) $R_2 - R_1$, $R_3 - 2R_1$: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 2 \\ 0 & -2 & 0 & 3 \end{array}\right]$, then $R_3 - R_2$: $\left[\begin{array}{ccc|c} \mathbf{1} & 1 & 1 & 1 \\ 0 & \mathbf{-2} & 0 & 2 \\ 0 & 0 & 0 & \mathbf{1} \end{array}\right]$. $\operatorname{rank} A = 2 < 3 = \operatorname{rank}[A \mid \vec{\mathbf{b}}]$: **no solution** (the last row says $0 = 1$).

(e) $\left[\begin{array}{cccc|c} \mathbf{1} & 1 & -1 & 1 & 2 \\ 0 & \mathbf{1} & 1 & -1 & 1 \end{array}\right]$ is already in REF; $R_1 - R_2$ gives $\left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & 1 \\ 0 & 1 & 1 & -1 & 1 \end{array}\right]$. $\operatorname{rank} A = 2 = \operatorname{rank}[A \mid \vec{\mathbf{b}}]$; $4 - 2 = 2$ free variables ($x_3 = s$, $x_4 = t$); dimension 2:

$$(x_1, x_2, x_3, x_4) = (1, 1, 0, 0) + s(2, -1, 1, 0) + t(-2, 1, 0, 1).$$

Common mistake

Counting the pivots of A in (d) as 3 because the matrix has three nonzero rows. The pivot of row 3 is in the *last* column: it counts for $\operatorname{rank}[A \mid \vec{\mathbf{b}}]$, not for $\operatorname{rank} A$.

Solution A12 THINK ⋆⋆ *Reasoning with the ranks only*

(a) The ranks are equal: consistent. Free variables: $n - \operatorname{rank} A = 3 - 3 = 0$. **Exactly one solution** (a point), even though there are more equations than unknowns.

(b) $[A \mid \vec{\mathbf{b}}]$ has only 3 rows, so $\operatorname{rank}[A \mid \vec{\mathbf{b}}] \leq 3$. It is also $\geq \operatorname{rank} A = 3$ (adding a column cannot remove pivots). So both ranks are 3, for every $\vec{\mathbf{b}}$: always consistent. Dimension $4 - 3 = 1$: a line in $\mathbb{R}^4$.

(c) Consistent, dimension $3 - 1 = 2$: a plane. With rank 1, the three equations are multiples of one another: they describe the **same plane**.

(d) $\operatorname{rank} A < \operatorname{rank}[A \mid \vec{\mathbf{b}}]$: no solution. $\operatorname{rank} A = 1$ means the second row of A is a multiple of the first: the normal vectors are parallel. The planes are **parallel and distinct**.

(e) **No.** $\operatorname{rank} A \leq 3$ (three rows) < 5, so there are at least $5 - 3 = 2$ free variables. The system has either no solution or infinitely many.

Takeaway. Two numbers decide everything: $\operatorname{rank} A$ versus $\operatorname{rank}[A \mid \vec{\mathbf{b}}]$ (existence), and $\operatorname{rank} A$ versus n (uniqueness, dimension).

Solution A13 COMPUTE ⋆⋆

Method. For a homogeneous system, the last column is all zeros and stays zero: reduce A alone. Then use "particular + homogeneous".

(a)
$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 5 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 + R_2} \begin{bmatrix} \mathbf{1} & 0 & 3 \\ 0 & \mathbf{1} & 1 \end{bmatrix}.$$

$z = t$: $x = -3t$, $y = -t$: $\vec{\mathbf{h}} = t(-3, -1, 1)$, $t \in \mathbb{R}$: a **line through the origin** (the intersection of two planes through O).

Check. $(1, -1, 2) \times (2, -1, 5) = (-5 + 2,\ 4 - 5,\ -1 + 2) = (-3, -1, 1)$ ✓.

(b) $3-2+0=1$ ✓ and $6-2+0=4$ ✓. The two systems have the same coefficients, so

$$(x,y,z)=\underbrace{(3,2,0)}_{\vec{\mathbf{p}}}+t\underbrace{(-3,-1,1)}_{\vec{\mathbf{h}}},\qquad t\in\mathbb{R}.$$

(c) By substitution: $0-1+2=1$ ✓ and $0-1+5=4$ ✓: **yes**. With (b): $(0,1,1)-(3,2,0)=(-3,-1,1)$, the homogeneous solution with $t=1$.

Takeaway. The solutions of $A\vec{\mathbf{x}}=\vec{\mathbf{b}}$ form the homogeneous line, **shifted** by $\vec{\mathbf{p}}$: two parallel lines. Once $A\vec{\mathbf{x}}=\vec{\mathbf{0}}$ is solved, one particular solution gives all the others.

Solution A14 COMPUTE ⋆⋆

Method. Swap first so that the pivot is 1 (not k), eliminate, factor the last pivot.

$$\left[\begin{array}{cc|c}k&4&2\\1&k&1\end{array}\right]\xrightarrow{R_1\leftrightarrow R_2}\left[\begin{array}{cc|c}\mathbf{1}&k&1\\k&4&2\end{array}\right]\xrightarrow{R_2\leftarrow R_2-kR_1}\left[\begin{array}{cc|c}1&k&1\\0&4-k^2&2-k\end{array}\right],\qquad 4-k^2=(2-k)(2+k).$$

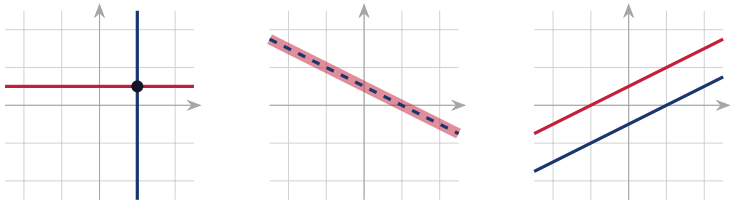
case	last row	solutions
$k\neq\pm2$	pivot $(2-k)(2+k)\neq0$	exactly one
$k=2$	$[\,0\;\;0\mid0\,]$	infinitely many (the line $x+2y=1$)
$k=-2$	$[\,0\;\;0\mid4\,]$	none

(a) For $k\neq\pm2$, we may divide by $(2-k)(2+k)$:

$$y=\frac{2-k}{(2-k)(2+k)}=\frac{1}{2+k},\qquad x=1-ky=\frac{2+k-k}{2+k}=\frac{2}{2+k}.$$

Check. $k=0$: the system is $4y=2$, $x=1$, so $(1,\frac12)$; the formulas give $(\frac22,\frac12)$ ✓.

(b)



$k=0$: one point $k=2$: same line $k=-2$: parallel

The normals are $(k,4)$ and $(1,k)$. They are parallel exactly when $k\cdot k-4\cdot1=0$, i.e. $k=\pm2$: the *same* expression as the last pivot. Then the lines are either equal ($k=2$: $2x+4y=2$ is twice $x+2y=1$) or parallel and distinct ($k=-2$: $-2x+4y=2$ and $x-2y=1$, i.e. $x-2y=-1$ and $x-2y=1$).

Solution A15 COMPUTE ⋆⋆

Method. Reduce $[\,A \mid I\,]$ with Gauss–Jordan, doing the same operations on both halves. A zero row on the left means "not invertible".

(a)

$$\left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 2 & 1 & -1 & 1 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{cc|cc} \mathbf{1} & 0 & 2 & -1 \\ 2 & 1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & \mathbf{1} & -5 & 3 \end{array}\right] \quad \Longrightarrow \quad A^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}.$$

The first two moves are not the "standard" ones: they produce a pivot equal to 1 without any fraction.

Comparison with the formula. $ad - bc = 6 - 5 = 1$, so $A^{-1} = \frac{1}{1}\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$: the same matrix.

Check. $AA^{-1} = \begin{bmatrix} 6-5 & -3+3 \\ 10-10 & -5+6 \end{bmatrix} = I$ ✓.

(b)

$$\left[\begin{array}{ccc|ccc} \mathbf{1} & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & \mathbf{1} & 1 & 0 & 1 & 0 \\ 0 & 0 & \mathbf{1} & -1 & 0 & 1 \end{array}\right]$$

Three pivots equal to 1 already. Going up:

$$\xrightarrow{R_2 \leftarrow R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \left[\begin{array}{ccc|ccc} \mathbf{1} & 0 & 0 & -1 & -2 & 2 \\ 0 & \mathbf{1} & 0 & 1 & 1 & -1 \\ 0 & 0 & \mathbf{1} & -1 & 0 & 1 \end{array}\right].$$

$$B^{-1} = \begin{bmatrix} -1 & -2 & 2 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$

Check. Row 1 of B · column 1 of B^{-1}: $-1 + 2 + 0 = 1$ ✓; row 3 of B · column 2: $-2 + 2 + 0 = 0$ ✓.

(c)

$$\left[\begin{array}{ccc|ccc} \mathbf{1} & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & \mathbf{1} & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right].$$

A zero row on the left: the left part can never become I, so C is **not invertible**. (Row 3 of C is the sum of rows 1 and 2.)

Solution A16 COMPUTE ⋆⋆ *Invertible or not? Use the rank*

Method. A square matrix of order n is invertible if and only if its rank is n. Reduce to an echelon form and count the pivots: no inverse is needed.

(a) Already in REF with pivots 1, 4, 6: rank 3, **invertible**. An upper triangular matrix with nonzero diagonal entries always has rank n.

(b) $R_2 - 4R_1$, $R_3 - 7R_1$: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}$, then $R_3 - 2R_2$ gives a zero row: rank $2 < 3$, **not invertible**.

(c) $R_3 - R_1$: $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$, then $R_3 + R_2$: $\begin{bmatrix} \mathbf{1} & 1 & 0 \\ 0 & \mathbf{1} & 1 \\ 0 & 0 & \mathbf{2} \end{bmatrix}$: rank 3, **invertible**.

(d) $R_1 \leftrightarrow R_2$: $\begin{bmatrix} \mathbf{1} & k \\ k & 4 \end{bmatrix}$ (pivot without parameter), then $R_2 \leftarrow R_2 - kR_1$: $\begin{bmatrix} 1 & k \\ 0 & 4 - k^2 \end{bmatrix}$. The rank is 2 iff $4 - k^2 \neq 0$: **invertible iff** $k \neq \pm 2$. These are exactly the values for which the system of Exercise A14 has **one** solution: a square system has a unique solution exactly when its coefficient matrix is invertible ($\vec{\mathbf{x}} = A^{-1}\vec{\mathbf{b}}$).

Takeaway. For a square matrix: rank $n \iff$ RREF $= I_n \iff$ invertible. A zero row in an echelon form is enough to conclude "not invertible".

Solution A17 COMPUTE ⋆ *Theatre tickets*

Method. Unknowns: a, s, c = numbers of adult, student and child tickets. One equation per piece of information.

$$\begin{cases} a + s + c = 500 & \text{(number of tickets)} \\ 15a + 10s + 6c = 5520 & \text{(revenue)} \\ a - s - c = -100 & \text{(“100 fewer adults”)} \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 500 \\ 15 & 10 & 6 & 5520 \\ 1 & -1 & -1 & -100 \end{array}\right]$$

$$\xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - 15R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 500 \\ 0 & -5 & -9 & -1980 \\ 0 & -2 & -2 & -600 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3,\ R_2 \leftarrow -\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 500 \\ 0 & \mathbf{1} & 1 & 300 \\ 0 & -5 & -9 & -1980 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 + 5R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 500 \\ 0 & 1 & 1 & 300 \\ 0 & 0 & \mathbf{-4} & -480 \end{array}\right].$$

$c = 120$, $s = 300 - 120 = 180$, $a = 500 - 300 = 200$.

Check. $200 + 180 + 120 = 500$ ✓, $3000 + 1800 + 720 = 5520$ ✓, $200 = 300 - 100$ ✓. The answers are whole and nonnegative, as they should be.

Part B — Going further

Solution B1 TRUE / FALSE ⋆⋆ *Solutions and row operations*

(a) **True.** A solution of the new system satisfies all its equations, in particular all the old ones, so it was already a solution. Adding an equation can only keep or remove solutions.

(b) **True.** $R_2 \leftarrow 3R_2 - R_1$ is $R_2 \leftarrow 3R_2$ (allowed, since $3 \neq 0$) followed by $R_2 \leftarrow R_2 - R_1$; it is undone by $R_2 \leftarrow \frac{1}{3}(R_2 + R_1)$.

(c) **False.** Every elementary operation can be undone, so nothing is lost. A zero row means that this equation was a *combination of the others*: it gave no new condition (Exercise A9(a)).

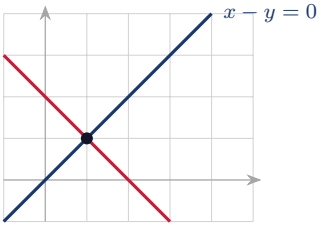
(d) **False.** The "prism" $x + y + z = 1$, $x + 2y + 3z = 2$, $2x + 3y + 4z = 4$: no two normals are parallel, yet there is no solution. The third normal is the sum of the first two, but the constants do not follow ($1 + 2 \neq 4$).

Solution B2 THINK ⋆⋆ *Spot the error*

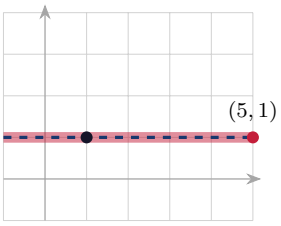
(a) Add the two equations: $2x = 2$, so $x = 1$, then $y = 2 - x = 1$. The unique solution is $(1, 1)$. In matrix form: $\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 0 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & -2 \end{array}\right]$, so $y = 1$ and $x = 1$.

(b) $(5, 1)$ in the student's system: $0 \cdot 5 + 2 \cdot 1 = 2$ ✓ and $0 \cdot 5 - 2 \cdot 1 = -2$ ✓. In the original: $5 + 1 = 6 \neq 2$ ×. So the student's system has *more* solutions than the original: it is not equivalent.

What went wrong. The two updates were done *simultaneously*, both using the *old* rows. Done correctly, the second update should use the *new* row 1, $(0, 2, 2)$: $R_2 - R_1^{\text{new}} = (1, -1, 0) - (0, 2, 2) = (1, -3, -2)$, and the system stays equivalent. With the student's simultaneous step, the new rows $(0, 2, 2)$ and $(0, -2, -2)$ are multiples of each other: *both* say $y = 1$, and the information about x has disappeared.



original: one point $(1, 1)$



student: both rows say $y = 1$

Common mistake

Writing several operations on the same arrow is fine *only* if they modify different rows using a row that does not change (e.g. $R_2 \leftarrow R_2 - 2R_1$ and $R_3 \leftarrow R_3 - R_1$). Never update two rows with each other in one step.

Solution B3 THINK ⋆⋆ *Two students, two answers?*

Student 1: $\xrightarrow{R_1 \leftarrow \frac{1}{3}R_1,\ R_2 \leftarrow \frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right]$.

Student 2: $\xrightarrow{R_2 \leftarrow -\frac{1}{6}R_2} \left[\begin{array}{cc|c} 1 & 4 & 7 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 - 4R_2} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right]$.

Both reach the same RREF, and the same solution $(x, y) = (-1, 2)$. This is not a coincidence: a matrix has only one reduced echelon form.

Takeaway. Two students who disagree on the REF can both be right; if they disagree on the RREF, one of them made a mistake.

Solution B4 THINK ⋆⋆ *The intersection of two planes*

(a)

$$\left[\begin{array}{ccc|c} \mathbf{1} & 2 & -1 & 3 \\ 2 & 3 & 1 & 4 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & \mathbf{-1} & 3 & -2 \end{array}\right].$$

$z = t$: $-y + 3t = -2 \Rightarrow y = 2 + 3t$; $x = 3 - 2y + z = 3 - 4 - 6t + t = -1 - 5t$.

$$\mathcal{P}_1 \cap \mathcal{P}_2 : \quad (x, y, z) = (-1, 2, 0) + t(-5, 3, 1), \qquad t \in \mathbb{R}.$$

Check. $(1, 2, -1) \times (2, 3, 1) = \big(2 + 3,\ -2 - 1,\ 3 - 4\big) = (5, -3, -1) = -(-5, 3, 1)$: parallel to our direction ✓.

(b) Substitute the line into $x + y + z = 0$:

$$(-1 - 5t) + (2 + 3t) + t = 0 \iff 1 - t = 0 \iff t = 1 : \qquad \text{the point } (-6, 5, 1).$$

Check. $-6 + 5 + 1 = 0$ ✓, and $(-6, 5, 1)$ is on $\mathcal{P}_1$: $-6 + 10 - 1 = 3$ ✓.

(c) A point on both planes satisfies *any combination* of their equations. For example the sum:

$$(x + 2y - z) + (2x + 3y + z) = 3 + 4 \iff 3x + 5y = 7.$$

This plane contains the whole line (check: $3(-1-5t)+5(2+3t) = 7$ for every t). Elimination would turn the third row into a zero row, because it is $R_1 + R_2$.

Takeaway. "The third equation is a combination of the first two" (algebra) and "the third plane contains the line of the first two" (geometry) are the same statement.

Solution B5 THINK ⋆⋆ *Build the system*

Method. Work backwards from the geometry: choose normal vectors with the right properties, then adjust the constants.

The line has direction $(1, 1, 0)$ and passes through $(1, 0, 2)$. Each plane containing it has a normal perpendicular to $(1, 1, 0)$; two non-parallel choices are $(1, -1, 0)$ and $(0, 0, 1)$. The constants come from the point:

$$\begin{cases} x - y = 1 - 0 = 1 \\ z = 2 \end{cases}$$

Check. For every t: $(1+t)-t=1$ ✓and $z=2$ ✓. Since the two normals are not parallel, the solution set is exactly one line, so it is our line. Many other answers are correct; check yours by elimination.

Solution B6 COMPUTE ⋆⋆ *Two parameters*

Method. The pivots 1 do not contain the parameters: eliminate, then discuss the last row.

$$\left[\begin{array}{ccc|c} \mathbf{1} & 1 & 2 & 1 \\ 2 & 3 & 5 & 4 \\ 1 & 2 & a & b \end{array}\right] \xrightarrow[R_3\leftarrow R_3-R_1]{R_2\leftarrow R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & \mathbf{1} & 1 & 2 \\ 0 & 1 & a-2 & b-1 \end{array}\right] \xrightarrow{R_3\leftarrow R_3-R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & \boldsymbol{a-3} & b-3 \end{array}\right].$$

case	last row	solutions
$a \neq 3$	pivot $a-3 \neq 0$	exactly one
$a=3,\ b\neq 3$	$[\,0\ 0\ 0 \mid b-3\neq 0\,]$	none
$a=3,\ b=3$	$[\,0\ 0\ 0 \mid 0\,]$	infinitely many (a line)

For $a \neq 3$: $z=\frac{b-3}{a-3}$, then $y=2-z$ and $x=1-y-2z=-1-z$.
For $a=b=3$: $z=t$, $y=2-t$, $x=1-y-2z=-1-t$:

$$(x,y,z)=(-1,2,0)+t(-1,-1,1), \qquad t\in\mathbb{R}.$$

Check. $(-1-t)+(2-t)+2t=1$ ✓, $2(-1-t)+3(2-t)+5t=4$ ✓, $(-1-t)+2(2-t)+3t=3$ ✓(for every t).

Solution B7 THINK ⋆⋆ *Spot the error*

Error. The operation $R_2 \leftarrow \frac{1}{k-1}R_2$ is an elementary operation only if $k-1\neq 0$. For $k=1$ the student divided by 0, and the special case disappeared.

Correct discussion. After $R_2 \leftarrow R_2 - R_1$, the matrix is $\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & k-1 & k-1 \end{array}\right]$.

- $k\neq 1$: divide by $k-1$: $y=1$, $x=1$. **Unique solution** $(1,1)$.
- $k=1$: the second row is $[\,0\ \ 0 \mid 0\,]$. The system reduces to $x+y=2$: **infinitely many solutions**, $(x,y)=(2-t,\,t)$.

Check. For $k=1$ the original system is $x+y=2$, $x+y=2$: the same line twice ✓.

Common mistake

Simplifying $\frac{k-1}{k-1}=1$ without a condition. Every division by an expression containing the parameter must come with "if this expression is not 0", and the other case must be studied separately.

Solution B8 COMPUTE ⋆⋆ *An invertible matrix?*

$$\begin{bmatrix} \mathbf{1} & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & k \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & \mathbf{1} & 1 \\ 0 & 1 & k-1 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & \boldsymbol{k-2} \end{bmatrix}.$$

Three pivots iff $k-2 \neq 0$: E **is invertible iff** $k \neq 2$ (invertible matrix theorem: the RREF is then I_3).
For $k=2$: $z=t$, $y+z=0 \Rightarrow y=-t$, $x+2y+z=0 \Rightarrow x=t$: $\quad \vec{\mathbf{x}} = (1,-1,1)$.
Check. $E(1,-1,1) = (1-2+1,\ 2-5+3,\ 1-3+2) = (0,0,0)$ ✓.

Solution B9 COMPUTE ⋆⋆ *A circle through three points*

Method. Each point (x_0, y_0) on the circle gives the condition $x_0^2 + y_0^2 + ax_0 + by_0 + c = 0$: a linear equation in a, b, c.

(a)

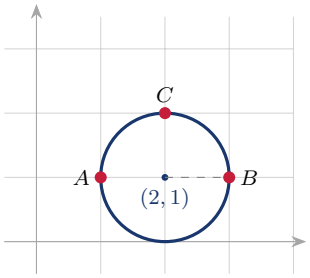
$$\begin{aligned} A(1,1): &\quad 1+1+a+b+c=0 &&\Rightarrow\ a+b+c=-2 \\ B(3,1): &\quad 9+1+3a+b+c=0 &&\Rightarrow\ 3a+b+c=-10 \\ C(2,2): &\quad 4+4+2a+2b+c=0 &&\Rightarrow\ 2a+2b+c=-8 \end{aligned}$$

$$\left[\begin{array}{ccc|c} \mathbf{1} & 1 & 1 & -2 \\ 3 & 1 & 1 & -10 \\ 2 & 2 & 1 & -8 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & \mathbf{-2} & -2 & -4 \\ 0 & 0 & \mathbf{-1} & -4 \end{array}\right]$$

$c=4$, $-2b-8=-4 \Rightarrow b=-2$, $a=-2-b-c=-4$. The circle is $x^2+y^2-4x-2y+4=0$. Completing the squares:

$$(x^2-4x+4)+(y^2-2y+1) = -4+4+1 \iff (x-2)^2+(y-1)^2 = 1:$$

centre $(2,1)$**, radius** 1.



Check. A, B, C are at distance 1 from $(2,1)$ ✓.

(b) The *unknowns* are a, b, c, not x, y. Once the point is chosen, x_0^2, y_0^2, x_0, y_0 are numbers, and a, b, c appear only to the first power. "Linear" always refers to the unknowns.

(c)

$$(0,0):\ c=0; \qquad (1,1):\ a+b+c=-2; \qquad (2,2):\ 2a+2b+c=-8.$$

With $c=0$: $a+b=-2$ and $2a+2b=-8$, i.e. $a+b=-4$. Contradiction: **no solution**. Geometrically, the three points are on the line $y=x$, and a line meets a circle in at most two points: no circle passes through three aligned points.

Solution B10 COMPUTE ⋆⋆ *Traffic in a city centre*

Method. At each intersection: (cars in) = (cars out). The arrows in the figure tell which flows come in and which go out.

(a)

$$\begin{aligned} A: &\quad 500 = x_1 + x_4 && \Rightarrow x_1 + x_4 = 500 \\ B: &\quad x_1 + 100 = x_2 + 350 && \Rightarrow x_1 - x_2 = 250 \\ C: &\quad x_2 + x_3 = 400 \\ D: &\quad x_4 + 150 = x_3 && \Rightarrow x_3 - x_4 = 150 \end{aligned}$$

$$\left[\begin{array}{cccc|c} \mathbf{1} & 0 & 0 & 1 & 500 \\ 1 & -1 & 0 & 0 & 250 \\ 0 & 1 & 1 & 0 & 400 \\ 0 & 0 & 1 & -1 & 150 \end{array}\right] \xrightarrow{R_2 \leftarrow R_1 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 500 \\ 0 & \mathbf{1} & 0 & 1 & 250 \\ 0 & 1 & 1 & 0 & 400 \\ 0 & 0 & 1 & -1 & 150 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 500 \\ 0 & 1 & 0 & 1 & 250 \\ 0 & 0 & \mathbf{1} & -1 & 150 \\ 0 & 0 & 1 & -1 & 150 \end{array}\right]$$

$$\xrightarrow{R_4 \leftarrow R_4 - R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 500 \\ 0 & 1 & 0 & 1 & 250 \\ 0 & 0 & 1 & -1 & 150 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

(The first operation is $R_2 \leftarrow -R_2$ followed by $R_2 \leftarrow R_2 + R_1$.) The zero row is expected: the total inflow 750 equals the total outflow 750, so one balance equation follows from the other three. With $x_4 = t$:

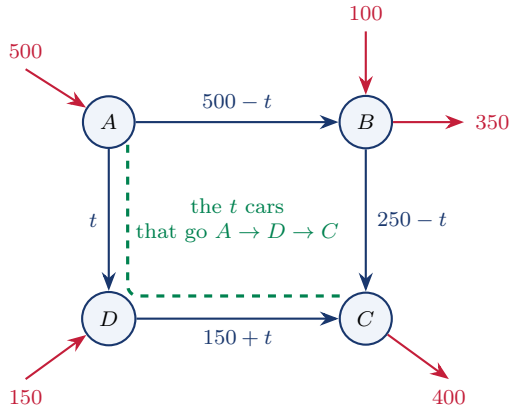
$$x_1 = 500 - t, \qquad x_2 = 250 - t, \qquad x_3 = 150 + t, \qquad x_4 = t.$$

(b) Increasing t by one means one more car per hour goes $A \to D \to C$ instead of $A \to B \to C$: x_4 and x_3 increase, x_1 and x_2 decrease. The balance at every intersection is unchanged.

(c) All flows ≥ 0: $t \geq 0$, $500 - t \geq 0$, $250 - t \geq 0$, $150 + t \geq 0$, so $0 \leq t \leq 250$:

$$250 \leq x_1 \leq 500, \qquad 0 \leq x_2 \leq 250, \qquad 150 \leq x_3 \leq 400, \qquad 0 \leq x_4 \leq 250.$$

(d) $x_2 = 250 - t \leq 200 \iff t \geq 50$: at least 50 cars per hour must use street AD (and then $50 \leq t \leq 250$).





Takeaway. The algebra gives infinitely many solutions; the situation (nonnegative flows, capacities) selects an *interval* of acceptable ones. The free variable has a concrete meaning: an alternative route.

Solution B11 PROVE ⋆⋆ *Two solutions are never alone*

Method. Compute $A(\dots)$ with the rules of the matrix chapter: $A(\vec{\mathbf{u}}+\vec{\mathbf{v}}) = A\vec{\mathbf{u}} + A\vec{\mathbf{v}}$ and $A(t\vec{\mathbf{u}}) = t\,A\vec{\mathbf{u}}$.

(a) $A(\vec{\mathbf{x}}_1 - \vec{\mathbf{x}}_2) = A\vec{\mathbf{x}}_1 - A\vec{\mathbf{x}}_2 = \vec{\mathbf{b}} - \vec{\mathbf{b}} = \vec{\mathbf{0}}$.

(b) By (a), $A(\vec{\mathbf{x}}_2 - \vec{\mathbf{x}}_1) = \vec{\mathbf{0}}$. So for every real t,

$$A\big(\vec{\mathbf{x}}_1 + t(\vec{\mathbf{x}}_2 - \vec{\mathbf{x}}_1)\big) = A\vec{\mathbf{x}}_1 + t\,A(\vec{\mathbf{x}}_2 - \vec{\mathbf{x}}_1) = \vec{\mathbf{b}} + t\,\vec{\mathbf{0}} = \vec{\mathbf{b}}.$$

Since $\vec{\mathbf{d}} = \vec{\mathbf{x}}_2 - \vec{\mathbf{x}}_1 \neq \vec{\mathbf{0}}$, different values of t give different points ($t\vec{\mathbf{d}} = s\vec{\mathbf{d}}$ forces $t = s$). So two solutions give infinitely many: a linear system never has exactly two solutions.

(c) $A(\vec{\mathbf{x}}_1 + \vec{\mathbf{x}}_2) = \vec{\mathbf{b}} + \vec{\mathbf{b}} = 2\vec{\mathbf{b}}$, and $2\vec{\mathbf{b}} = \vec{\mathbf{b}} \iff \vec{\mathbf{b}} = \vec{\mathbf{0}}$.

Takeaway. Solution − solution = homogeneous solution; solution + homogeneous solution = solution. The synthesis exercise that follows uses exactly these two facts.

Solution B12 THINK ⋆⋆⋆ *Synthesis: two solutions, many consequences*

Method. Use Exercise B11: the difference of two solutions solves $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$, and adding it (times any t) to a solution gives another solution.

(a) The line through the two solutions consists of solutions:

$$(1,2,0) + t\big((3,1,1) - (1,2,0)\big) = (1,2,0) + t(2,-1,1), \qquad t \in \mathbb{R}.$$

(b) $A(2,-1,1) = A\big((3,1,1) - (1,2,0)\big) = A(3,1,1) - A(1,2,0) = \vec{\mathbf{b}} - \vec{\mathbf{b}} = \vec{\mathbf{0}}$.

(c) **No**: by (b), $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has the nontrivial solution $(2,-1,1)$, and the invertible matrix theorem says that an invertible matrix has only the trivial solution. (Directly: if A^{-1} existed, $(2,-1,1) = A^{-1}A(2,-1,1) = A^{-1}\vec{\mathbf{0}} = \vec{\mathbf{0}}$, which is false.)

(d) **No**. If $\vec{\mathbf{q}}$ solves $A\vec{\mathbf{x}} = \vec{\mathbf{c}}$, then $A\big(\vec{\mathbf{q}} + t(2,-1,1)\big) = \vec{\mathbf{c}} + t\vec{\mathbf{0}} = \vec{\mathbf{c}}$ for every t: infinitely many solutions. So for every $\vec{\mathbf{c}}$, the system $A\vec{\mathbf{x}} = \vec{\mathbf{c}}$ has either no solution or infinitely many, never exactly one.

Takeaway. One piece of information about one system (two different solutions) determines the behaviour of *every* system with the same matrix A. The number of solutions depends on A (through its homogeneous solutions) far more than on $\vec{\mathbf{b}}$.

What to remember from Chapter 6

- Elementary row operations never change the solution set, because each one can be undone.
- Read the echelon form in this order: first look for a contradiction $[\,0 \cdots 0 \mid c \neq 0\,]$, then count the free variables.
- Every answer has a geometric meaning: empty, a point, a line or a plane, read from the normal vectors.
- Solutions of $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ = one particular solution + solutions of $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$.
- $[\,A \mid I\,] \to [\,I \mid A^{-1}\,]$; a zero row on the left means "not invertible".
- In applications, a free variable usually has a concrete meaning, and the situation (signs, integers, capacities) restricts it.