Chapter 6 – Summary

Systems of linear equations: echelon forms, free variables and rank

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How to use this summary

Each section gives the definitions, worked examples and a picture. The **red boxes** are the traps that cost the most points on exams. Sections 3 to 5 are the heart of the chapter: echelon forms, free variables, and the rank, which tells you at once whether a system has solutions and what the solution set looks like. Finish with the self-check on the last page *before* reading the answers.

1 Linear systems and their solutions

Linear system

A system of m linear equations in n unknowns $x_1, \dots, x_n$: each unknown appears to the first power, multiplied by a constant,

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \qquad (i = 1, \dots, m).$$

In matrix form: $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$, with A of size $m \times n$. A **solution** satisfies **all** the equations at once.

The fundamental fact: 0, 1 or ∞

A linear system has **no solution**, **exactly one**, or **infinitely many**, never exactly two. Reason: if $\vec{\mathbf{p}} \neq \vec{\mathbf{q}}$ are solutions, then $A\big(\vec{\mathbf{p}} + t(\vec{\mathbf{q}} - \vec{\mathbf{p}})\big) = \vec{\mathbf{b}} + t(\vec{\mathbf{b}} - \vec{\mathbf{b}}) = \vec{\mathbf{b}}$ for every t: the whole line through $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ is made of solutions.

Two solutions are never alone

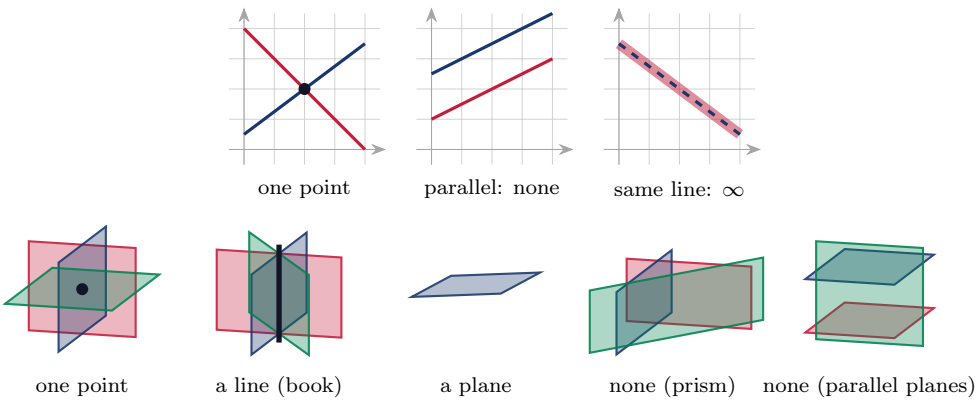
$(1, 2, 3)$ and $(3, 2, 1)$ solve $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$. Then every point of the line through them is a solution:

$$(1, 2, 3) + t\big((3, 2, 1) - (1, 2, 3)\big) = (1 + 2t,\ 2,\ 3 - 2t),$$

for instance $t = \frac{1}{2}$: $(2, 2, 2)$; $t = 2$: $(5, 2, -1)$; $t = -1$: $(-1, 2, 5)$.

But the *sum* $(4, 4, 4)$ gives $A(4, 4, 4) = \vec{\mathbf{b}} + \vec{\mathbf{b}} = 2\vec{\mathbf{b}}$: it is a solution **only if** $\vec{\mathbf{b}} = \vec{\mathbf{0}}$.

Geometry: in $\mathbb{R}^2$ each equation is a line; in $\mathbb{R}^3$ each equation $ax + by + cz = d$ is a plane with normal vector (a, b, c)



2 The augmented matrix and row operations

System $\longleftrightarrow$ augmented matrix

$$\begin{cases} x+y+z=2 \\ 2x+3y+z=7 \\ x-y+2z=-3 \end{cases} \longleftrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 7 \\ 1 & -1 & 2 & -3 \end{array}\right]$$

One row per equation, one column per unknown, constants after the bar; a missing unknown is a 0. Put every equation in the form $ax+by+cz=d$ first: $y+3=x$ becomes $[\,-1 \;\; 1 \;\; 0 \,|\, -3\,]$.

The three elementary row operations

swap	$R_i \leftrightarrow R_j$	undo: itself
scale	$R_i \leftarrow kR_i,\ k \neq 0$	undo: $\frac{1}{k}$
replace	$R_i \leftarrow R_i + kR_j,\ j \neq i$	undo: $-k$

Each one can be **undone**, so the new system is **equivalent** to the old one: same solutions.

Not allowed

$R_2 \leftarrow 0 \cdot R_2$, $R_2 \leftarrow R_2 - R_2$, $R_2 \leftarrow 2R_1$ (the old R_2 is lost), multiplying a row by an unknown. Do one operation at a time, always with the *current* rows; two rows must never be updated with each other in the same step.

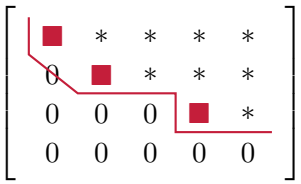
3 Row echelon form and Gaussian elimination

Row echelon form (REF)

A matrix is in **row echelon form** if

1. the zero rows (if any) are at the **bottom**;
2. the first nonzero entry of each nonzero row, called its **pivot**, is strictly to the **right** of the pivot of the row above.

Consequence: every entry **below** a pivot is 0. In a REF, pivots do *not* have to be 1, and entries *above* a pivot can be anything.



■ = pivot $\neq 0$, $*$ = any number.
Each step down moves **at least** one column right; a step may skip a column (here column 3).

In REF (pivots in red)

$$E_1 = \begin{bmatrix} \mathbf{2} & -1 & 3 \\ 0 & \mathbf{5} & 1 \\ 0 & 0 & \mathbf{-4} \end{bmatrix} \quad E_2 = \begin{bmatrix} \mathbf{1} & 3 & 0 & 2 \\ 0 & 0 & \mathbf{6} & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad E_3 = \begin{bmatrix} 0 & \mathbf{3} & 1 & 4 \\ 0 & 0 & 0 & \mathbf{2} \end{bmatrix} \quad E_4 = \begin{bmatrix} \mathbf{1} & 2 & -1 & 4 \end{bmatrix}$$

E_2: the step from row 1 to row 2 skips column 2. E_3: a first column of zeros is allowed. E_4: a single nonzero row is always in REF.

Not in REF, and one operation that fixes it

$$N_1 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 5 & 6 \end{bmatrix} \quad N_2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix} \quad N_3 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad N_4 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 1 & 0 \end{bmatrix}$$

N_1: the pivot of row 3 is *left* of the pivot of row 2 ($R_2 \leftrightarrow R_3$). N_2: zero row not at the bottom ($R_2 \leftrightarrow R_3$). N_3: nonzero entry below the first pivot ($R_2 \leftarrow R_2 - 2R_1$). N_4: two rows start in the same column ($R_3 \leftarrow R_3 - R_2$).

Gaussian elimination

1. Leftmost nonzero column: put a nonzero entry on top (swap if needed). It is the **pivot**.
2. Replacements: **zeros below** the pivot.
3. Cover the pivot row; repeat below.
4. **Back substitution**, from the bottom up.

Tip: pivots equal to ± 1 avoid fractions.

$x + y + z = 2,\ \ 2x + 3y + z = 7,\ \ x - y + 2z = -3$

$$\left[\begin{array}{ccc|c} \mathbf{1} & 1 & 1 & 2 \\ 2 & 3 & 1 & 7 \\ 1 & -1 & 2 & -3 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & \mathbf{1} & -1 & 3 \\ 0 & -2 & 1 & -5 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 + 2R_2} \left[\begin{array}{ccc|c} \mathbf{1} & 1 & 1 & 2 \\ 0 & \mathbf{1} & -1 & 3 \\ 0 & 0 & \mathbf{-1} & 1 \end{array}\right] \quad \text{(REF)}$$

Bottom up: $-z = 1 \Rightarrow z = -1$; $y - z = 3 \Rightarrow y = 2$; $x + y + z = 2 \Rightarrow x = 1$.
Check in the *original* system: $1+2-1 = 2$, $2+6-1 = 7$, $1 - 2 - 2 = -3$ ✓

Avoiding fractions: $x+y+2z=9,\ \ 2x+4y-3z=1,\ \ 3x+6y-5z=0$

$$\left[\begin{array}{ccc|c} \mathbf{1} & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right] \xrightarrow[R_3\leftarrow R_3-3R_1]{R_2\leftarrow R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & \mathbf{2} & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right] \xrightarrow{R_3\leftarrow 2R_3-3R_2} \left[\begin{array}{ccc|c} \mathbf{1} & 1 & 2 & 9 \\ 0 & \mathbf{2} & -7 & -17 \\ 0 & 0 & \mathbf{-1} & -3 \end{array}\right]$$

($R_3 \leftarrow 2R_3 - 3R_2$ is a scaling by $2 \neq 0$ followed by a replacement: allowed.) Bottom up: $z = 3$, $2y - 21 = -17 \Rightarrow y = 2$, $x + 2 + 6 = 9 \Rightarrow x = 1$. Check: $1+2+6=9$, $2+8-9=1$, $3+12-15=0$ ✓

4 Reduced row echelon form and Gauss–Jordan

Reduced row echelon form (RREF)

A matrix is in **RREF** if it is in REF and moreover

3. every pivot is equal to **1**;
4. every pivot is the **only** nonzero entry of its column (zeros *above* it too).

A matrix has **many** echelon forms but **exactly one** reduced echelon form.

$$\left[\begin{array}{cccc} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{array}\right] \longrightarrow \left[\begin{array}{cccc} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$$

REF RREF

In RREF or not?

$$\begin{bmatrix} \mathbf{1} & 0 & 3 \\ 0 & \mathbf{1} & -2 \end{bmatrix}$$ yes $$\begin{bmatrix} \mathbf{1} & 2 & 0 & 1 \\ 0 & 0 & \mathbf{1} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ yes $$\begin{bmatrix} \mathbf{1} & 3 & 2 \\ 0 & \mathbf{1} & 4 \end{bmatrix}$$ no $$\begin{bmatrix} \mathbf{2} & 0 & 6 \\ 0 & \mathbf{1} & -1 \end{bmatrix}$$ no

Second matrix: the 2 in column 2 is allowed, since column 2 has *no* pivot. Third: a 3 sits above a pivot ($R_1 \leftarrow R_1 - 3R_2$ fixes it). Fourth: a pivot equals 2 ($R_1 \leftarrow \frac{1}{2}R_1$ fixes it).

Gauss–Jordan: continue the example upwards (pivots to 1, then clear above them, from the last pivot)

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & \mathbf{-1} & 1 \end{array}\right] \xrightarrow{R_3\leftarrow -R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & \mathbf{1} & -1 \end{array}\right] \xrightarrow[R_1\leftarrow R_1-R_3]{R_2\leftarrow R_2+R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & \mathbf{1} & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{R_1\leftarrow R_1-R_2} \left[\begin{array}{ccc|c} \mathbf{1} & 0 & 0 & 1 \\ 0 & \mathbf{1} & 0 & 2 \\ 0 & 0 & \mathbf{1} & -1 \end{array}\right]$$

The RREF **displays** the solution $x = 1$, $y = 2$, $z = -1$: no back substitution is needed.

Two REFs, one RREF $\quad M = \left[\begin{array}{cc|c} 2 & 4 & 6 \\ 1 & 3 & 5 \end{array}\right]$

Student 1: $\left[\begin{array}{cc|c} 2 & 4 & 6 \\ 0 & 1 & 2 \end{array}\right]$ (by $R_2 \leftarrow R_2 - \frac{1}{2}R_1$). Student 2: $\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -2 & -4 \end{array}\right]$ (by $R_1 \leftrightarrow R_2$, then $R_2 \leftarrow R_2 - 2R_1$).

Two different echelon forms, both correct. Both reduce to the **same** RREF $\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right]$, i.e. $(x, y) = (-1, 2)$.

5 Free variables, rank and the solution set

5.1 Which variables are free?

Pivot variables and free variables

Reduce $[\,A \mid \vec{\mathbf{b}}\,]$ to an echelon form and look **only at the columns of** A (left of the bar):

- a column **with** a pivot: its unknown is a **pivot variable**;
- a column **without** a pivot: its unknown is a **free variable**.

Method. (1) Check that there is no row $[\,0\cdots0 \mid c\,]$ with $c \neq 0$. (2) Give each free variable a parameter $s, t, \ldots$ (3) Each row of the RREF expresses its pivot variable in terms of the free ones.

Unknowns x_1, x_2, x_3, x_4

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 4 \\ 2 & 4 & 1 & 5 & 10 \\ 1 & 2 & 1 & 2 & 6 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_2} \left[\begin{array}{cccc|c} \mathbf{1} & 2 & 0 & 3 & 4 \\ 0 & 0 & \mathbf{1} & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \text{(RREF)}$$

Pivots in columns 1 and 3: x_1, x_3 are pivot variables; $x_2 = s$ and $x_4 = t$ are free. The rows read $x_1 + 2x_2 + 3x_4 = 4$ and $x_3 - x_4 = 2$:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 - 2s - 3t \\ s \\ 2 + t \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \qquad s, t \in \mathbb{R}.$$

Reading solutions straight from an RREF (unknowns x, y, z)

$\left[\begin{array}{ccc\|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$	pivots in columns 1, 2; $z = t$ free	$(x, y, z) = (3, 1, 0) + t(-2, 1, 1)$: a line
$\left[\begin{array}{ccc\|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{array}\right]$	pivots in columns 1, 3; the *middle* $y = s$ is free	$(x, y, z) = (0, 0, 5) + s(-2, 1, 0)$: a line
$\left[\begin{array}{ccc\|c} 1 & -1 & 3 & 2 \end{array}\right]$	pivot in column 1; $y = s$, $z = t$ free	$(x, y, z) = (2, 0, 0) + s(1, 1, 0) + t(-3, 0, 1)$: a plane

Warning

- The free variables are **not** always the last ones: above, x_2 (or y) is free and x_3 (or z) is not.
- The bar column never gives a variable: a pivot there means **no solution**.
- A REF is enough to see which columns have pivots; the RREF is the fastest way to write the solutions.

5.2 The rank

Rank of a matrix

The **rank** of a matrix M, written $\operatorname{rank} M$, is the **number of pivots** in an echelon form of M, i.e. the number of **nonzero rows** of that echelon form.
All echelon forms of M have the same number of pivots (they lead to the same RREF), so the rank does not depend on the computation.

For a system $A\vec{x} = \vec{b}$

- $\operatorname{rank} A$: number of pivots in the **coefficient** columns;
- $\operatorname{rank}[\,A \mid \vec{b}\,]$: number of pivots in **all** the columns, the last one included.

Reduce $[\,A \mid \vec{b}\,]$ only **once**: its left part is then an echelon form of A.

Same A, two right-hand sides

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$. After $R_2 \leftarrow R_2 - R_1$, $R_3 \leftarrow R_3 - 2R_1$, $R_3 \leftarrow R_3 - R_2$:

$$\vec{b} = (1,2,4): \left[\begin{array}{ccc|c} \mathbf{1} & 1 & 1 & 1 \\ 0 & \mathbf{1} & 2 & 1 \\ 0 & 0 & 0 & \mathbf{1} \end{array}\right]$$

$\operatorname{rank} A = 2$, $\operatorname{rank}[\,A \mid \vec{b}\,] = 3$.

$$\vec{b} = (1,2,3): \left[\begin{array}{ccc|c} \mathbf{1} & 1 & 1 & 1 \\ 0 & \mathbf{1} & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\operatorname{rank} A = 2$, $\operatorname{rank}[\,A \mid \vec{b}\,] = 2$.

Bounds

At most one pivot per row and per column: $\operatorname{rank} A \le m$, $\operatorname{rank} A \le n$, and

$$\operatorname{rank} A \le \operatorname{rank}[\,A \mid \vec{b}\,] \le \operatorname{rank} A + 1.$$

Ranks at a glance

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$: rank 1 (row 2 $= 2\times$ row 1). $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$: rank 2 ($R_1 - 2R_2 + R_3 = \vec{0}$). $\begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \\ 3 & -3 & 6 \end{bmatrix}$: rank 1 (every row is a multiple of $(1,-1,2)$).

5.3 The rank theorem: existence, free variables, dimension

Theorem (A **is** $m \times n$, $r = \operatorname{rank} A$)

1. **Existence.** $A\vec{x} = \vec{b}$ is consistent $\iff$ $\operatorname{rank} A = \operatorname{rank}[\,A \mid \vec{b}\,]$. If $\operatorname{rank}[\,A \mid \vec{b}\,] = r + 1$, the extra pivot is in the last column: a row $[\,0 \cdots 0 \mid c\,]$ with $c \neq 0$, i.e. $0 = c$.
2. **Number of free variables.** There are exactly $n - r$ free variables (unknowns minus pivots).
3. **Dimension of the solution set.** If the system is consistent, its solution set is described by $n - r$ parameters: it has **dimension** $n - r$.
4. **Uniqueness.** Exactly one solution $\iff$ consistent and $r = n$.

Dimension = number of parameters (consistent case)

$n-r$	solution set	in $\mathbb{R}^3$
0	$\vec{\mathbf{p}}$	a point
1	$\vec{\mathbf{p}}+t\vec{\mathbf{u}}$	a line
2	$\vec{\mathbf{p}}+s\vec{\mathbf{u}}+t\vec{\mathbf{v}}$	a plane
3	all of $\mathbb{R}^3$	($A=O$, $\vec{\mathbf{b}}=\vec{\mathbf{0}}$)

Back to the example of 5.2 ($n=3$)

$\vec{\mathbf{b}}=(1,2,4)$: $\operatorname{rank} A = 2 \neq 3 = \operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,]$: **no solution** (prism).
$\vec{\mathbf{b}}=(1,2,3)$: $2=2$, so consistent, with $3-2=1$ free variable $z=t$: a **line**,

$$(x,y,z)=(0,1,0)+t(1,-2,1).$$

In the example of 5.1: $n=4$, $r=2$: dimension 2, a "plane" in $\mathbb{R}^4$.

One system, the right-hand side decides: $x+2y-z=1,\ 2x+5y+z=7,\ x+y-4z=c$

$$\left[\begin{array}{ccc|c}1&2&-1&1\\2&5&1&7\\1&1&-4&c\end{array}\right] \xrightarrow[R_3\leftarrow R_3-R_1]{R_2\leftarrow R_2-2R_1} \left[\begin{array}{ccc|c}1&2&-1&1\\0&1&3&5\\0&-1&-3&c-1\end{array}\right] \xrightarrow{R_3\leftarrow R_3+R_2} \left[\begin{array}{ccc|c}\mathbf{1}&2&-1&1\\0&\mathbf{1}&3&5\\0&0&0&c+4\end{array}\right]$$

$c=-4$: $\operatorname{rank} A = 2 = \operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,]$, $3-2=1$ free variable: a **line** $(-9,5,0)+t(7,-3,1)$.
$c\neq -4$: last row $[\,0\ 0\ 0 \mid c+4\,]$ with $c+4\neq 0$: $\operatorname{rank} A = 2 < 3 = \operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,]$: **no solution**.

Five systems in $\mathbb{R}^3$, read with the ranks

echelon form of $[\,A \mid \vec{\mathbf{b}}\,]$	$\operatorname{rank} A$	$\operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,]$	free var.	dim.	geometry
$\left[\begin{array}{ccc\|c}\mathbf{1}&1&1&2\\0&\mathbf{1}&-1&3\\0&0&\mathbf{-1}&1\end{array}\right]$	3	3	0	0	one point
$\left[\begin{array}{ccc\|c}\mathbf{1}&0&-1&0\\0&\mathbf{1}&2&1\\0&0&0&0\end{array}\right]$	2	2	1 (z)	1	a line (book)
$\left[\begin{array}{ccc\|c}\mathbf{1}&2&-1&4\\0&0&0&0\\0&0&0&0\end{array}\right]$	1	1	2 (y,z)	2	a plane (same plane 3×)
$\left[\begin{array}{ccc\|c}\mathbf{1}&1&1&1\\0&\mathbf{1}&2&1\\0&0&0&\mathbf{1}\end{array}\right]$	2	3	—	—	none (prism)
$\left[\begin{array}{ccc\|c}\mathbf{1}&1&1&1\\0&0&0&\mathbf{1}\end{array}\right]$	1	2	—	—	none (two parallel planes)

Warning

A zero row does **not** mean infinitely many solutions: four planes through one point give a 4×4 augmented matrix with a zero row and $\operatorname{rank} A = 3 = n$, so exactly one solution. Always compare r with n.

Consequences

- $m < n$ (more unknowns than equations): $r \leq m < n$, so a consistent system has infinitely many solutions. It can still be inconsistent ($x + y + z = 1$, $x + y + z = 2$).
- $m > n$ (more equations than unknowns): every outcome is possible; only the comparison of $\operatorname{rank} A$ with $\operatorname{rank}[\, A \mid \vec{\mathbf{b}}\,]$ decides. Example: $x + y = 2$, $x - y = 0$, $3x + y = 4$ has ranks $2 = 2 = n$: exactly one solution $(1, 1)$.
- $m = n$ (square): $r = n$ gives exactly one solution for *every* $\vec{\mathbf{b}}$; $r < n$ gives, for each $\vec{\mathbf{b}}$, none or infinitely many.
- Whether a solution *exists* depends on $\vec{\mathbf{b}}$; once one exists, the dimension $n - \operatorname{rank} A$ depends on A alone.

6 Homogeneous systems

Homogeneous system $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$

Always consistent ($\vec{\mathbf{0}}$ is a solution, and $\operatorname{rank} A = \operatorname{rank}[\, A \mid \vec{\mathbf{0}}\,]$).

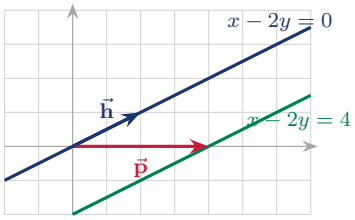
- $\operatorname{rank} A = n$: only the **trivial** solution $\vec{\mathbf{0}}$;
- $\operatorname{rank} A < n$: infinitely many, a set of dimension $n - \operatorname{rank} A$ through the origin. In particular when $m < n$.

Structure of the solutions

If $\vec{\mathbf{p}}$ is **one** solution of $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$, all the solutions are $\vec{\mathbf{x}} = \vec{\mathbf{p}} + \vec{\mathbf{h}}$ with $A\vec{\mathbf{h}} = \vec{\mathbf{0}}$: the homogeneous solution set, shifted by $\vec{\mathbf{p}}$ (same dimension).

$x - 2y + z = 0, \quad 2x - 3y + 4z = 0$

RREF $\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \end{bmatrix}$: $r = 2$, $n = 3$, a line through O: $\vec{\mathbf{h}} = t(-5, -2, 1)$.
With right-hand side $(2, 5)$, $\vec{\mathbf{p}} = (4, 1, 0)$ is a solution, so all solutions are $(4, 1, 0) + t(-5, -2, 1)$.



In $\mathbb{R}^2$: same direction, shifted by $\vec{\mathbf{p}}$.

Particular + homogeneous in $\mathbb{R}^4$

$x_1 + x_2 - x_3 + 2x_4 = 0$, $\quad x_3 - x_4 = 0$: $\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 + R_2} \begin{bmatrix} \mathbf{1} & 1 & 0 & 1 \\ 0 & 0 & \mathbf{1} & -1 \end{bmatrix}$, $\operatorname{rank} A = 2$, dimension $4 - 2 = 2$: $\vec{\mathbf{h}} = s(-1, 1, 0, 0) + t(-1, 0, 1, 1)$.
With right-hand side $(2, 0)$: $\vec{\mathbf{p}} = (1, 0, 1, 1)$ works ($1 + 0 - 1 + 2 = 2$, $\quad 1 - 1 = 0$), so all solutions are $\vec{\mathbf{x}} = (1, 0, 1, 1) + s(-1, 1, 0, 0) + t(-1, 0, 1, 1)$.

7 Systems with a parameter

Method

Eliminate with pivots that do **not** contain the parameter, factor what ends up in pivot position, then split into cases and compute $\operatorname{rank} A$ and $\operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,]$ in each case. Never divide by an expression that may be 0.

$kx + 4y = 2, \quad x + ky = 1$

$$\left[\begin{array}{cc|c} k & 4 & 2 \\ 1 & k & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & k & 1 \\ k & 4 & 2 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - kR_1} \left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & (2-k)(2+k) & 2-k \end{array}\right]$$

case	$\operatorname{rank} A$	$\operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,]$	solutions
$k \neq \pm 2$	2	2	exactly one: $\left(\frac{2}{k+2}, \frac{1}{k+2}\right)$
$k = 2$	1	1	a line (one parameter): the two equations describe the same line
$k = -2$	1	2	none: two parallel lines

Two parameters: $x + 2y + z = 1, \quad 2x + 5y + 3z = 3, \quad x + 3y + kz = m$

$$\xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & k-1 & m-1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_2} \left[\begin{array}{ccc|c} \mathbf{1} & 2 & 1 & 1 \\ 0 & \mathbf{1} & 1 & 1 \\ 0 & 0 & k-2 & m-2 \end{array}\right]$$

case	$\operatorname{rank} A$	$\operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,]$	solutions
$k \neq 2$	3	3	one point: $z = \frac{m-2}{k-2}, \quad y = 1 - z, \quad x = z - 1$
$k = 2,\ m \neq 2$	2	3	none
$k = 2,\ m = 2$	2	2	a line: $(-1, 1, 0) + t(1, -1, 1)$

The classic trap

$R_2 \leftarrow \frac{1}{k-2} R_2$ is an elementary operation **only if** $k \neq 2$. The case $k = 2$ must be studied separately, from the matrix *before* the division.

8 Computing the inverse with Gauss–Jordan

$B = \begin{bmatrix}1&1&2\\2&3&3\\1&2&2\end{bmatrix}$: $[\,B \mid I\,] \to [\,I \mid B^{-1}\,]$

$$\left[\begin{array}{ccc|ccc}1&1&2&1&0&0\\2&3&3&0&1&0\\1&2&2&0&0&1\end{array}\right] \xrightarrow[\text{then } R_3\leftarrow R_3-R_2]{R_2-2R_1,\ R_3-R_1} \left[\begin{array}{ccc|ccc}\mathbf{1}&1&2&1&0&0\\0&\mathbf{1}&-1&-2&1&0\\0&0&\mathbf{1}&1&-1&1\end{array}\right]$$

$$\xrightarrow[\text{then } R_1\leftarrow R_1-R_2]{R_2+R_3,\ R_1-2R_3} \left[\begin{array}{ccc|ccc}1&0&0&0&2&-3\\0&1&0&-1&0&1\\0&0&1&1&-1&1\end{array}\right]$$

$B^{-1} = \begin{bmatrix}0&2&-3\\-1&0&1\\1&-1&1\end{bmatrix}$. Check one entry of BB^{-1}: row $1 \cdot$ column $1 = 0 - 1 + 2 = 1$ ✓. Why it works: $[\,B \mid I\,]$ solves the n systems $B\vec{\mathbf{x}} = \vec{\mathbf{e}}_1, \ldots, B\vec{\mathbf{x}} = \vec{\mathbf{e}}_n$ at once.

Invertible matrix theorem (A is $n \times n$)

A invertible $\iff$ $\operatorname{rank} A = n$ $\iff$ $\operatorname{RREF}(A) = I_n$ $\iff$ $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has exactly one solution for every $\vec{\mathbf{b}}$ $\iff$ $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has only $\vec{\mathbf{0}}$. If a zero row appears on the left of $[\,A \mid I\,]$, then $\operatorname{rank} A < n$ and A is not invertible.

Invertible or not, with a parameter: $D = \begin{bmatrix}1&1&1\\1&2&3\\1&3&k\end{bmatrix}$

$$D \xrightarrow[R_3\leftarrow R_3-R_1]{R_2\leftarrow R_2-R_1} \begin{bmatrix}1&1&1\\0&1&2\\0&2&k-1\end{bmatrix} \xrightarrow{R_3\leftarrow R_3-2R_2} \begin{bmatrix}\mathbf{1}&1&1\\0&\mathbf{1}&2\\0&0&k-5\end{bmatrix}$$

$k \neq 5$: $\operatorname{rank} D = 3$, **invertible**. $k = 5$: $\operatorname{rank} D = 2 < 3$, not invertible; $z = t$, $y = -2t$, $x = t$ gives the nonzero solution $(1, -2, 1)$ of $D\vec{\mathbf{x}} = \vec{\mathbf{0}}$ (check: $1 - 2 + 1 = 0$, $1 - 4 + 3 = 0$, $1 - 6 + 5 = 0$ ✓).

9 Cheat sheet

Notion	What to remember
REF	zero rows at the bottom; each pivot strictly right of the one above
RREF	REF + pivots equal to 1 + zeros above and below each pivot; unique
pivot / free variable	column of A with / without a pivot (never the bar column)
$\operatorname{rank} M$	number of pivots (= nonzero rows) of an echelon form of M
consistent	$\operatorname{rank} A = \operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,]$ (otherwise a row $0 = c \neq 0$)
free variables	$n - \operatorname{rank} A$
dimension of the solution set	$n - \operatorname{rank} A$ (point 0, line 1, plane 2), when consistent
unique solution	consistent and $\operatorname{rank} A = n$
A^{-1}	$[\,A \mid I\,] \to [\,I \mid A^{-1}\,]$; exists $\iff \operatorname{rank} A = n$

Order of the questions

1. Is there a row $0 = c \neq 0$? 2. If not, how many free variables?

Always

Check a solution in the **original** system, not in the echelon form.

10 Self-check

True or false? Justify in one line: a general reason if true, one counterexample if false.

1. A matrix has only one row echelon form.
2. If $\operatorname{rank} A = \operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,]$, the system has exactly one solution.
3. $\operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,]$ is always equal to $\operatorname{rank} A$ or to $\operatorname{rank} A + 1$.
4. A 3×5 matrix has rank at most 3.
5. The free variables are always the last unknowns.
6. If the echelon form of $[\,A \mid \vec{\mathbf{b}}\,]$ has a zero row, there are infinitely many solutions.
7. A homogeneous system with $\operatorname{rank} A < n$ has a nontrivial solution.
8. A consistent system of 2 equations in 4 unknowns has a solution set of dimension at least 2.

Answers

1. **False**: $M = \left[\begin{array}{cc|c} 2 & 4 & 6 \\ 1 & 3 & 5 \end{array}\right]$ has (at least) two echelon forms (Section 4); only the RREF is unique.
2. **False**: equal ranks only mean *consistent*. With $r < n$ there are infinitely many ($x + y = 1$: $r = 1 < 2$).
3. **True**: adding one column can add at most one pivot.
4. **True**: at most one pivot per row.
5. **False**: in $\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right]$, x_2 is free and x_3 is not.
6. **False**: four planes through one point give a zero row and $\operatorname{rank} A = 3 = n$: one solution.
7. **True**: there are $n - \operatorname{rank} A \geq 1$ free variables; give one of them the value 1.
8. **True**: $\operatorname{rank} A \leq 2$, so the dimension is $4 - \operatorname{rank} A \geq 2$.