Review — Chapters 5 and 6

Matrices · Systems of linear equations

Mohamed Nouidha

How this series is organised

Part A tests Chapter 5 (matrices), **Part B** tests Chapter 6 (systems of linear equations), and **Part C** mixes the two chapters. Tags: COMPUTE a calculation, TRUE / FALSE decide and justify (a general argument if true, one explicit counterexample if false), THINK reasoning. Stars give the difficulty. Check every answer you can: a product by multiplying back, a solution in the *original* system.

Part A — Matrices

Exercise A1 COMPUTE ⋆ *Building and classifying*

(a) Write the 3×3 matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ defined by $a_{ij} = (-1)^{i+j}(i + j)$ and $b_{ij} = i - j$. Is A symmetric? antisymmetric? Same questions for B. Justify from the rules, without looking at the matrices.

(b) For each matrix, say which words apply: square, diagonal, scalar, upper triangular, lower triangular, symmetric, antisymmetric.

$$M_1 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \quad M_2 = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 4 \\ 1 & -4 & 0 \end{bmatrix} \quad M_3 = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 0 & -1 & 5 \end{bmatrix}$$

$$M_4 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 0 & 0 \end{bmatrix} \quad M_5 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Exercise A2 COMPUTE ⋆ *Equality*

Find a, b and c so that $S = \begin{bmatrix} 1 & a+b & 3 \\ 4 & 2 & c \\ a & 5 & 0 \end{bmatrix}$ is symmetric. Can you choose a, b, c so that S is antisymmetric?

Exercise A3 COMPUTE ⋆ *Operations and sizes*

Let $A = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 3 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 3 & 1 \end{bmatrix}$.

(a) Compute $2A - 3B$.

(b) Compute AC and CA. Compare their sizes and their traces.

(c) Compute $(AC)^T$ and $C^T A^T$.

(d) Which of these are defined: $A + C$, AB, $A^T B$, BC, C^2, CC^T? Give the size of each one that is, and compute $A^T B$.

Exercise A4 COMPUTE ⋆⋆ *Matrix equations*

Solve each equation for X. Isolate X first, compute at the end.

(a) $2(X-P)+Q^T=3X$, where $P=\begin{bmatrix}1&0\\2&-1\end{bmatrix}$ and $Q=\begin{bmatrix}3&4\\-2&1\end{bmatrix}$. Check your answer.

With inverses. In (b)–(g), A, B, C are square matrices of the same size, and every inverse you need exists. Express X in terms of A, B, C (and I). Watch the side on which you multiply.

(b) $AX+B=C$

(c) $XA-B=C$

(d) $AX=X+B$

(e) $XA+X=B$

(f) $(AX)^{-1}=B$

(g) $AX^TB^{-1}=C$

With numbers. Let $M=\begin{bmatrix}2&1\\3&2\end{bmatrix}$ and $N=\begin{bmatrix}1&2\\-1&0\end{bmatrix}$. Solve for X and check each answer.

(h) $MX=N$

(i) $XM=N$ (compare with (h))

(j) $MX-N=4X$

(k) $(X+N)^{-1}=M$

Exercise A5 COMPUTE ⋆⋆ *Columns*

Let $A=\begin{bmatrix}\vec{\mathbf{a}}_1&\vec{\mathbf{a}}_2&\vec{\mathbf{a}}_3\end{bmatrix}$ with $\vec{\mathbf{a}}_1=\begin{bmatrix}1\\0\\2\end{bmatrix}$, $\vec{\mathbf{a}}_2=\begin{bmatrix}-1\\1\\0\end{bmatrix}$, $\vec{\mathbf{a}}_3=\begin{bmatrix}0\\3\\1\end{bmatrix}$.

(a) Compute $A\begin{bmatrix}2\\1\\-1\end{bmatrix}$ as a combination of the columns of A, then by the row-times-column rule.

(b) Write the system $\begin{cases}x-y=1\\y+3z=-2\\2x+z=4\end{cases}$ as one matrix equation, and as one vector equation using $\vec{\mathbf{a}}_1,\vec{\mathbf{a}}_2,\vec{\mathbf{a}}_3$.

(c) Which vector $\vec{\mathbf{x}}$ gives $A\vec{\mathbf{x}}=\vec{\mathbf{a}}_2$? And $A\vec{\mathbf{x}}=\vec{\mathbf{a}}_1+\vec{\mathbf{a}}_3$?

Exercise A6 COMPUTE ⋆⋆ *Powers and trace*

(a) Let $N=\begin{bmatrix}0&2&1\\0&0&3\\0&0&0\end{bmatrix}$. Compute N^2 and N^3.

(b) Explain why $(I+N)^2=I+2N+N^2$, then compute $(I+N)^2$.

(c) Let $M = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$. Compute M^2, $\operatorname{tr} M$ and $\operatorname{tr}(M^2)$.

Exercise A7 COMPUTE ⋆⋆ *Inverses*

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

(a) Compute A^{-1}, B^{-1}, and $(AB)^{-1}$ in two different ways.

(b) Solve for X: $AXB = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$. Check your answer.

(c) For which values of k is $\begin{bmatrix} k & 2 \\ 2k & k \end{bmatrix}$ invertible?

Exercise A8 TRUE / FALSE ⋆⋆ *Matrices*

(a) If A and B are 2×2 and $AB = O$, then $A = O$ or $B = O$.

(b) $(A + B)(A - B) = A^2 - B^2$ for all square matrices A, B of the same size.

(c) If A is symmetric, then A^2 is symmetric.

(d) If A is 2×3, then $A^T A$ is a symmetric 3×3 matrix.

(e) $\operatorname{tr}(AB) = \operatorname{tr}(A)\ \operatorname{tr}(B)$ for all 2×2 matrices.

(f) The product of two upper triangular 3×3 matrices is upper triangular.

Part B — Systems of linear equations

Exercise B1 COMPUTE ⋆

Solve by Gaussian elimination, then check in the original system:

$$\begin{cases} y + 2z = 5 \\ x + y + z = 4 \\ 2x + 3y - z = -2 \end{cases}$$

Exercise B2 COMPUTE ⋆⋆

Reduce the augmented matrix to RREF, give the pivot variables and the free variables, and write the solution set in parametric form (unknowns x_1, x_2, x_3, x_4):

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 2 & 4 & -1 & 4 & 8 \\ 1 & 2 & 0 & 3 & 5 \end{array}\right]$$

Exercise B3 COMPUTE ⋆⋆ *Ranks*

For each system, give $\operatorname{rank} A$, $\operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,]$, the number of free variables and the dimension of the solution set; then solve it.

(a) $\begin{cases} x+y+z=2 \\ x+2y+3z=5 \\ 2x+3y+4z=8 \end{cases}$ (b) $\begin{cases} x-y=1 \\ 2x+y=5 \\ x+2y=4 \end{cases}$

(c) $\begin{cases} 2x-y+z=1 \\ 4x-2y+2z=2 \end{cases}$ (d) $\begin{cases} x_1+x_2+x_3+x_4=1 \\ x_1-x_2+x_3-x_4=3 \end{cases}$

Exercise B4 THINK ⋆⋆ *Reasoning with the ranks only*

(a) A is 5×3 and $\operatorname{rank} A = \operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,] = 3$. Describe the solution set of $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

(b) A is 2×4 with $\operatorname{rank} A = 2$. Is $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ consistent for every $\vec{\mathbf{b}}$? What is the dimension of its solution set?

(c) A is 4×4 with $\operatorname{rank} A = 3$. List every possible number of solutions of $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

(d) Can a system of 4 equations in 3 unknowns have infinitely many solutions?

(e) A is 3×3 and $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has no solution for some $\vec{\mathbf{b}}$. What can you say about the solutions of $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$?

Exercise B5 COMPUTE ⋆ *The rank theorem*

Each matrix is an echelon form of the augmented matrix $[\,A \mid \vec{\mathbf{b}}\,]$ of a system. Without solving the system, give for each one: the number of unknowns n, $\operatorname{rank} A$, $\operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,]$, whether the system is consistent, the free variables, the dimension of the solution set, and the number of solutions.

(a) $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 5 & 0 \end{array}\right]$ (b) $\left[\begin{array}{cccc|c} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 & 2 \end{array}\right]$ (c) $\left[\begin{array}{ccc|c} 2 & 1 & 0 & 4 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$

(d) $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ (e) $\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & 0 \end{array}\right]$ (f) $\left[\begin{array}{ccc|c} 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(The unknowns are x, y in (e), x, y, z in (a), (c), (d), (f), and x_1, x_2, x_3, x_4 in (b).)

Exercise B6 TRUE / FALSE ⋆⋆ *The rank theorem*

A is an $m\times n$ matrix.

(a) If $\operatorname{rank} A = \operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,]$, then $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has exactly one solution.

(b) If $\operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,] = \operatorname{rank} A + 1$, then $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has no solution.

(c) If A is 3×5 and $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is consistent, then it has at least two free variables.

(d) If the solution set of a system in 3 unknowns is a line, then $\operatorname{rank} A = 2$.

(e) If A is 4×4 and $\operatorname{rank} A = 4$, then $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has exactly one solution for every $\vec{\mathbf{b}}$.

(f) If the solution set of $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is a plane, then the solution set of $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ is also a plane.

Exercise B7 COMPUTE ⋆⋆ *Homogeneous and non-homogeneous*

(a) Solve $\begin{cases} x + y - z = 0 \\ 2x + 3y + z = 0 \end{cases}$ and describe its solution set geometrically.

(b) Check that $\vec{\mathbf{p}} = (-2, 3, 0)$ is a solution of $\begin{cases} x + y - z = 1 \\ 2x + 3y + z = 5 \end{cases}$, and write all its solutions without a new elimination.

(c) Is $(6, -3, 2)$ a solution of the system in (b)? Answer with the formula of (b), then by substitution.

Exercise B8 COMPUTE ⋆⋆ *Parameters*

(a) Discuss, according to a and b, the number of solutions of $\begin{cases} x + y + z = 1 \\ x + 2y + az = 2 \\ 2x + 3y + 4z = b \end{cases}$. Give the solution set in each case where it is not empty.

(b) For which values of k does $\begin{cases} kx + y + z = 0 \\ x + ky + z = 0 \\ x + y + kz = 0 \end{cases}$ have a nonzero solution? For each such value, give all the solutions and describe them geometrically.

Exercise B9 COMPUTE ⋆⋆ *The inverse with Gauss–Jordan*

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 3 & 4 \end{bmatrix}$.

(a) Compute A^{-1} with $[\, A \mid I \,]$. Check two entries of AA^{-1}.

(b) Use A^{-1} to solve $A\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $A\vec{\mathbf{x}} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

(c) What is $\operatorname{rank} A$? Justify without a new computation.

Exercise B10 COMPUTE ⋆ *Coffee sales*

A café sells small, medium and large coffees at \$2, \$3 and \$4. One morning it sold 120 coffees for a total of \$340, and it sold as many medium coffees as small and large coffees together. Name the unknowns, write the system, and find how many coffees of each size were sold. Check your answer in the statement.

Exercise B11 COMPUTE ⋆⋆ *Balancing a chemical equation*

Propane burns according to $a\,C_3H_8 + b\,O_2 \longrightarrow c\,CO_2 + d\,H_2O$.

(a) Write one equation per kind of atom (C, H, O). What kind of system do you obtain?

(b) Give the rank of the system and the number of free variables, then solve it.

(c) Give the smallest positive whole-number coefficients, and explain what the free variable means for the chemical equation.

Exercise B12 TRUE / FALSE ⋆⋆ *Systems*

(a) A system with more equations than unknowns has no solution.

(b) If A is 4×3 and $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has only the trivial solution, then $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has at most one solution for every $\vec{\mathbf{b}}$.

(c) If $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has infinitely many solutions, then $A\vec{\mathbf{x}} = \vec{\mathbf{c}}$ has infinitely many solutions for every $\vec{\mathbf{c}}$.

(d) Two systems whose augmented matrices have the same RREF have the same solutions.

(e) Swapping two columns of A does not change the solution set of $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

(f) If $\operatorname{rank} A$ equals the number of rows of A, then $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is consistent for every $\vec{\mathbf{b}}$.

Part C — Both chapters

Exercise C1 COMPUTE ⋆⋆ *A matrix equation $AX = B$*

Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ 2 & 0 \\ 1 & 2 \end{bmatrix}$.

(a) What is the size of a matrix X such that $AX = B$?

(b) Explain why the columns of X are the solutions of two systems with the same coefficient matrix.

(c) Find X by reducing $[\,A \mid B\,]$ once, then check by computing AX.

Exercise C2 TRUE / FALSE ⋆⋆ *Both chapters*

(a) If A is 3×3 with $\operatorname{rank} A = 3$, then A^T is invertible.

(b) If $AB = O$ and A is invertible, then $B = O$.

(c) If A is 2×3, then $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has a nonzero solution.

(d) If $A^2 = I$, then $A = I$ or $A = -I$.

(e) The system $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is consistent for every $\vec{\mathbf{b}}$.

(f) If A is invertible, then A^TA is invertible.