Review — Chapters 5 and 6

Detailed solutions

Mohamed Nouidha

Part A — Matrices

Exercise A1 COMPUTE ⋆ *Building and classifying*

(a) Substitute each position (i, j) in the rule:

$$A = \begin{bmatrix} 2 & -3 & 4 \\ -3 & 4 & -5 \\ 4 & -5 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}.$$

For instance $a_{12} = (-1)^3(3) = -3$, $a_{33} = (-1)^6(6) = 6$, $b_{31} = 3 - 1 = 2$.

A **is symmetric:** the rule only depends on $i + j$, and $i + j = j + i$, so $a_{ji} = (-1)^{j+i}(j + i) = a_{ij}$. A **is not antisymmetric:** an antisymmetric matrix has a zero diagonal, but $a_{11} = 2 \neq 0$.

B **is antisymmetric:** $b_{ji} = j - i = -(i - j) = -b_{ij}$ for all i, j. B **is not symmetric:** $b_{12} = -1$ but $b_{21} = 1$.

(b)
- M_1: square, diagonal, scalar ($3I_2$), upper triangular, lower triangular, symmetric. Not antisymmetric (nonzero diagonal).
- M_2: square, antisymmetric ($m_{21} = -m_{12} = -2$, $m_{31} = -m_{13} = 1$, $m_{32} = -m_{23} = -4$, zero diagonal). Not symmetric, not triangular, not diagonal.
- M_3: square, lower triangular (every entry above the diagonal is 0). Not upper triangular ($m_{21} = 4 \neq 0$), not diagonal, not symmetric ($m_{21} = 4 \neq m_{12} = 0$).
- M_4: 3×2, **not square**: none of the words apply (diagonal, triangular, symmetric and antisymmetric are defined for square matrices only).
- M_5: square, diagonal, hence upper and lower triangular, and symmetric. Not scalar (the diagonal entries $2, -1, 0$ are not all equal), not antisymmetric.

Exercise A2 COMPUTE ⋆ *Equality*

S is symmetric when $S^T = S$, that is $s_{ji} = s_{ij}$ for each pair of positions off the diagonal:

$$s_{12} = s_{21} : \ a + b = 4, \qquad s_{13} = s_{31} : \ 3 = a, \qquad s_{23} = s_{32} : \ c = 5.$$

So $a = 3$, $b = 1$, $c = 5$, and $S = \begin{bmatrix} 1 & 4 & 3 \\ 4 & 2 & 5 \\ 3 & 5 & 0 \end{bmatrix}$.

Antisymmetric: impossible. An antisymmetric matrix satisfies $s_{ii} = -s_{ii}$, so its diagonal is zero; here $s_{11} = 1$ and $s_{22} = 2$ do not depend on a, b, c.

Exercise A3 COMPUTE ⋆ *Operations and sizes*

(a) $2A-3B=\begin{bmatrix}2&-4&0\\6&2&8\end{bmatrix}-\begin{bmatrix}6&0&-3\\3&9&6\end{bmatrix}=\begin{bmatrix}-4&-4&3\\3&-7&2\end{bmatrix}$.

(b) A is 2×3 and C is 3×2, so AC is 2×2 and CA is 3×3.

$$AC=\begin{bmatrix}1+0+0&2+2+0\\3+0+12&6-1+4\end{bmatrix}=\begin{bmatrix}1&4\\15&9\end{bmatrix},\qquad CA=\begin{bmatrix}7&0&8\\-3&-1&-4\\6&-5&4\end{bmatrix}.$$

For example $(CA)_{11}=(1)(1)+(2)(3)=7$ and $(CA)_{32}=(3)(-2)+(1)(1)=-5$.
Different sizes, so $AC\neq CA$; but $\operatorname{tr}(AC)=1+9=10$ and $\operatorname{tr}(CA)=7-1+4=10$: the traces are equal, as $\operatorname{tr}(AC)=\operatorname{tr}(CA)$ always.

(c) $(AC)^T=\begin{bmatrix}1&15\\4&9\end{bmatrix}$. With $C^T=\begin{bmatrix}1&0&3\\2&-1&1\end{bmatrix}$ and $A^T=\begin{bmatrix}1&3\\-2&1\\0&4\end{bmatrix}$:

$$C^TA^T=\begin{bmatrix}1+0+0&3+0+12\\2+2+0&6-1+4\end{bmatrix}=\begin{bmatrix}1&15\\4&9\end{bmatrix}=(AC)^T,$$

as predicted by the rule $(AC)^T=C^TA^T$.

(d)
- $A+C$: **not defined** (2×3 and 3×2 are different sizes).
- AB: **not defined** ($(2\times 3)(2\times 3)$: $3\neq 2$).
- A^TB: $(3\times 2)(2\times 3)$, defined, 3×3.
- BC: $(2\times 3)(3\times 2)$, defined, 2×2.
- C^2: **not defined** (C is not square).
- CC^T: $(3\times 2)(2\times 3)$, defined, 3×3.

$$A^TB=\begin{bmatrix}1&3\\-2&1\\0&4\end{bmatrix}\begin{bmatrix}2&0&-1\\1&3&2\end{bmatrix}=\begin{bmatrix}2+3&0+9&-1+6\\-4+1&0+3&2+2\\0+4&0+12&0+8\end{bmatrix}=\begin{bmatrix}5&9&5\\-3&3&4\\4&12&8\end{bmatrix}.$$

Exercise A4 COMPUTE ⋆⋆ *Matrix equations*

(a) Isolate X first, compute at the end:

$$2X-2P+Q^T=3X\iff X=Q^T-2P.$$

$$X=\begin{bmatrix}3&-2\\4&1\end{bmatrix}-\begin{bmatrix}2&0\\4&-2\end{bmatrix}=\begin{bmatrix}1&-2\\0&3\end{bmatrix}.$$

Check: $2(X-P)+Q^T=2\begin{bmatrix}0&-2\\-2&4\end{bmatrix}+\begin{bmatrix}3&-2\\4&1\end{bmatrix}=\begin{bmatrix}3&-6\\0&9\end{bmatrix}=3X$ ✓

With inverses. The rule: to cancel a factor, multiply by its inverse **on the same side** as that factor, on **both** sides of the equation.

(b) $AX = C - B$. A is on the left of X, so multiply on the left by A^{-1}: $X = A^{-1}(C - B)$.

(c) $XA = B + C$. A is on the right of X, so multiply on the right by A^{-1}: $X = (B + C)A^{-1}$.

(d) $AX - X = B$, and $AX - X = AX - IX = (A - I)X$: $(A - I)X = B$, $X = (A - I)^{-1}B$.

(e) $XA + X = XA + XI = X(A + I)$, so $X(A + I) = B$: $X = B(A + I)^{-1}$.

Common mistake

Write $AX - X = (A - I)X$, **not** $(A - 1)X$: "$A - 1$" (a matrix minus a number) is not defined. And X stays on the side where it was: $XA + X = X(A + I)$, not $(A + I)X$.

(f) Take the inverse of both sides: $AX = \left((AX)^{-1}\right)^{-1} = B^{-1}$. Then $X = A^{-1}B^{-1}$, which is also $(BA)^{-1}$.
Check: $AX = AA^{-1}B^{-1} = B^{-1}$, so $(AX)^{-1} = (B^{-1})^{-1} = B$ ✓

(g) Left by A^{-1}, right by B: $X^T = A^{-1}CB$. Transpose: $X = (A^{-1}CB)^T = B^TC^T(A^{-1})^T$. (Also correct: $B^TC^T(A^T)^{-1}$, since $(A^{-1})^T = (A^T)^{-1}$.)

With numbers. $\det M = 4 - 3 = 1$, so $M^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$.

(h) $X = M^{-1}N = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2+1 & 4+0 \\ -3-2 & -6+0 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -5 & -6 \end{bmatrix}$.
Check: $MX = \begin{bmatrix} 6-5 & 8-6 \\ 9-10 & 12-12 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = N$ ✓

(i) $X = NM^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2-6 & -1+4 \\ -2+0 & 1+0 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ -2 & 1 \end{bmatrix}$.
Check: $XM = \begin{bmatrix} -8+9 & -4+6 \\ -4+3 & -2+2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = N$ ✓
Not the same X as in (h): $M^{-1}N \neq NM^{-1}$ because the product is not commutative.

(j) $MX - 4X = N$, so $(M - 4I)X = N$ with $M - 4I = \begin{bmatrix} -2 & 1 \\ 3 & -2 \end{bmatrix}$, $\det(M - 4I) = 4 - 3 = 1$, $(M - 4I)^{-1} = \begin{bmatrix} -2 & -1 \\ -3 & -2 \end{bmatrix}$:

$$X = (M - 4I)^{-1}N = \begin{bmatrix} -2 & -1 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -2+1 & -4+0 \\ -3+2 & -6+0 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ -1 & -6 \end{bmatrix}.$$

Check: $MX = \begin{bmatrix} -3 & -14 \\ -5 & -24 \end{bmatrix}$, $MX - N = \begin{bmatrix} -4 & -16 \\ -4 & -24 \end{bmatrix} = 4X$ ✓

(k) Take the inverse of both sides: $X + N = M^{-1}$, so

$$X = M^{-1} - N = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}.$$

Check: $X + N = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = M^{-1}$, whose inverse is M ✓

Exercise A5 COMPUTE ⋆⋆ *Columns*

Here $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 3 \\ 2 & 0 & 1 \end{bmatrix}$.

(a) As a combination of the columns:

$$A \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = 2\vec{\mathbf{a}}_1 + 1\vec{\mathbf{a}}_2 - 1\vec{\mathbf{a}}_3 = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}.$$

Row times column: $\big(2-1+0,\ 0+1-3,\ 4+0-1\big) = (1,-2,3)$. Same result.

(b) The rows of the system are exactly the rows of A:

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \qquad \text{and} \qquad x\,\vec{\mathbf{a}}_1 + y\,\vec{\mathbf{a}}_2 + z\,\vec{\mathbf{a}}_3 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}.$$

(c) $A\vec{\mathbf{x}} = x_1\vec{\mathbf{a}}_1 + x_2\vec{\mathbf{a}}_2 + x_3\vec{\mathbf{a}}_3$, so $\vec{\mathbf{x}} = (0,1,0)$ gives $\vec{\mathbf{a}}_2$, and $\vec{\mathbf{x}} = (1,0,1)$ gives $\vec{\mathbf{a}}_1 + \vec{\mathbf{a}}_3$.

Exercise A6 COMPUTE ⋆⋆ *Powers and trace*

(a) Row times column:

$$N^2 = \begin{bmatrix} 0 & 0 & 2\cdot 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad N^3 = N^2 N = O.$$

For N^3: the only nonzero row of N^2 is $(0,0,6)$, and the third row of N is zero, so every entry of N^2N is 0.

(b) $(I+N)^2 = (I+N)(I+N) = I^2 + IN + NI + N^2$. Since $IN = NI = N$, this is $I + 2N + N^2$ (the formula is correct here because I and N commute):

$$(I+N)^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 4 & 2 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 8 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}.$$

Check by multiplying $I+N = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ by itself: row 1 gives $(1,\ 2{+}2,\ 1{+}6{+}1) = (1,4,8)$ ✓

(c) $M^2 = \begin{bmatrix} 4-1 & 2+3 \\ -2-3 & -1+9 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -5 & 8 \end{bmatrix}$, $\quad \operatorname{tr} M = 2+3 = 5, \quad \operatorname{tr}(M^2) = 3+8 = 11.$

Note: $\operatorname{tr}(M^2) = 11 \neq (\operatorname{tr} M)^2 = 25$.

Exercise A7 COMPUTE ⋆⋆ *Inverses*

(a) $\det A = 2-1 = 1$ and $\det B = 1-0 = 1$:

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \qquad B^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}.$$

First way: $AB = \begin{bmatrix} 2 & 4+1 \\ 1 & 2+1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, $\det(AB) = 6-5 = 1$, so $(AB)^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.

Second way: $(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1+2 & -1-4 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.

Same result.

(b) Multiply on the **left** by A^{-1} and on the **right** by B^{-1}: $X = A^{-1}\begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}B^{-1}$.

$$A^{-1}\begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 4-2 \\ -1 & -4+4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}, \qquad X = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}.$$

Check: $AX = \begin{bmatrix} 2-1 & 0+2 \\ 1-1 & 0+2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$, then $(AX)B = \begin{bmatrix} 1 & 2+2 \\ 0 & 0+2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$ ✓

Common mistake

$X = A^{-1}CB^{-1}$, **not** $A^{-1}B^{-1}C$ or $(AB)^{-1}C$: each inverse stays on the side of the matrix it cancels.

(c) $ad - bc = k \cdot k - 2 \cdot 2k = k^2 - 4k = k(k-4)$. The matrix is invertible exactly when $k \neq 0$ and $k \neq 4$.

Exercise A8 TRUE / FALSE ⋆⋆ *Matrices*

(a) **False.** $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2-2 & -4+4 \\ 4-4 & -8+8 \end{bmatrix} = O$, with both factors nonzero.

(b) **False.** $(A+B)(A-B) = A^2 - AB + BA - B^2$, which equals $A^2 - B^2$ only when $AB = BA$. Counterexample: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $(A+B)(A-B) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, while $A^2 - B^2 = A - O = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(c) **True.** $(A^2)^T = (AA)^T = A^TA^T = AA = A^2$, using $(XY)^T = Y^TX^T$ and $A^T = A$.

(d) **True.** A^T is 3×2 and A is 2×3, so A^TA is 3×3; and $(A^TA)^T = A^T(A^T)^T = A^TA$.

(e) **False.** With $A = B = I_2$: $\operatorname{tr}(I_2I_2) = \operatorname{tr}(I_2) = 2$, while $\operatorname{tr}(I_2)\operatorname{tr}(I_2) = 2 \cdot 2 = 4$.

(f) **True.** Let A, B be upper triangular and $i > j$ (a position below the diagonal). In $(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j}$, each term $a_{ik}b_{kj}$ is 0: if $k < i$ then $a_{ik} = 0$, and if $k \geq i$ then $k > j$, so $b_{kj} = 0$. Hence $(AB)_{ij} = 0$ below the diagonal.

Part B — Systems of linear equations

Exercise B1 COMPUTE ⋆

The entry in position $(1,1)$ is 0, so we swap first:

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 5 \\ 1 & 1 & 1 & 4 \\ 2 & 3 & -1 & -2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} \mathbf{1} & 1 & 1 & 4 \\ 0 & 1 & 2 & 5 \\ 2 & 3 & -1 & -2 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & \mathbf{1} & 2 & 5 \\ 0 & 1 & -3 & -10 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_2} \left[\begin{array}{ccc|c} \mathbf{1} & 1 & 1 & 4 \\ 0 & \mathbf{1} & 2 & 5 \\ 0 & 0 & \mathbf{-5} & -15 \end{array}\right]$$

Back substitution: $-5z = -15 \Rightarrow z = 3$; $y + 2(3) = 5 \Rightarrow y = -1$; $x - 1 + 3 = 4 \Rightarrow x = 2$.

$$(x, y, z) = (2, -1, 3).$$

Check: $-1 + 6 = 5$ ✓, $2 - 1 + 3 = 4$ ✓, $4 - 3 - 3 = -2$ ✓

Exercise B2 COMPUTE ⋆⋆

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 2 & 4 & -1 & 4 & 8 \\ 1 & 2 & 0 & 3 & 5 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 + R_2} \left[\begin{array}{cccc|c} \mathbf{1} & 2 & 0 & 3 & 5 \\ 0 & 0 & \mathbf{1} & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This is the RREF. Pivots in columns 1 and 3: **pivot variables** x_1, x_3; **free variables** $x_2 = s$, $x_4 = t$. The rows read $x_1 + 2x_2 + 3x_4 = 5$ and $x_3 + 2x_4 = 2$:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 - 2s - 3t \\ s \\ 2 - 2t \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \qquad s, t \in \mathbb{R}.$$

Check with $s = t = 0$, i.e. $(5, 0, 2, 0)$: $5 - 2 = 3$ ✓, $10 - 2 = 8$ ✓, $5 = 5$ ✓. (And each direction vector gives 0 in the three left-hand sides, e.g. $(-3, 0, -2, 1)$: $-3 + 2 + 1 = 0$, $-6 + 2 + 4 = 0$, $-3 + 3 = 0$.)

Exercise B3 COMPUTE ⋆⋆ *Ranks*

(a)

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 5 \\ 2 & 3 & 4 & 8 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 4 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_2} \left[\begin{array}{ccc|c} \mathbf{1} & 1 & 1 & 2 \\ 0 & \mathbf{1} & 2 & 3 \\ 0 & 0 & 0 & \mathbf{1} \end{array}\right]$$

$\operatorname{rank} A = 2$, $\operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,] = 3$: the last row reads $0 = 1$, so the system is **inconsistent**: **no solution**. Free variables and dimension do not apply (the solution set is empty).

(b)
$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 2 & 1 & 5 \\ 1 & 2 & 4 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 3 & 3 \\ 0 & 3 & 3 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_2} \left[\begin{array}{cc|c} \mathbf{1} & -1 & 1 \\ 0 & \mathbf{3} & 3 \\ 0 & 0 & 0 \end{array}\right]$$

$\operatorname{rank} A = \operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,] = 2 = n$: consistent, 0 free variables, dimension 0: **exactly one solution**. $3y = 3 \Rightarrow y = 1$, $x = 1 + y = 2$: $(x, y) = (2, 1)$. **Check:** $2 - 1 = 1$, $4 + 1 = 5$, $2 + 2 = 4$ ✓ (three equations, two unknowns, and still exactly one solution).

(c)
$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 4 & -2 & 2 & 2 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{ccc|c} \mathbf{2} & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\operatorname{rank} A = \operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,] = 1$, $n = 3$: consistent, $3 - 1 = 2$ free variables, dimension 2: a **plane** (both equations describe the same plane). With $y = s$, $z = t$: $2x = 1 + s - t$, so

$$(x, y, z) = \left(\tfrac{1}{2} + \tfrac{1}{2}s - \tfrac{1}{2}t,\ s,\ t\right) = \left(\tfrac{1}{2}, 0, 0\right) + s\left(\tfrac{1}{2}, 1, 0\right) + t\left(-\tfrac{1}{2}, 0, 1\right), \qquad s, t \in \mathbb{R}.$$

(d)
$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 3 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 & 2 \end{array}\right] \xrightarrow{R_2 \leftarrow -\frac{1}{2}R_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 \end{array}\right]$$
$$\xrightarrow{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{cccc|c} \mathbf{1} & 0 & 1 & 0 & 2 \\ 0 & \mathbf{1} & 0 & 1 & -1 \end{array}\right]$$

$\operatorname{rank} A = \operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,] = 2$, $n = 4$: consistent, $4 - 2 = 2$ free variables ($x_3 = s$, $x_4 = t$), dimension 2:

$$\vec{\mathbf{x}} = (2 - s,\ -1 - t,\ s,\ t) = (2, -1, 0, 0) + s(-1, 0, 1, 0) + t(0, -1, 0, 1), \qquad s, t \in \mathbb{R}.$$

Check with $(2, -1, 0, 0)$: $2 - 1 = 1$ ✓, $2 + 1 = 3$ ✓

Exercise B4 THINK ⋆⋆ *Reasoning with the ranks only*

(a) The system is consistent ($\operatorname{rank} A = \operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,]$) and has $n - \operatorname{rank} A = 3 - 3 = 0$ free variables: **exactly one solution**, a single point of $\mathbb{R}^3$.

(b) Yes. $[\,A \mid \vec{\mathbf{b}}\,]$ has only 2 rows, so $\operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,] \le 2$; and $\operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,] \ge \operatorname{rank} A = 2$. So $\operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,] = 2 = \operatorname{rank} A$ for every $\vec{\mathbf{b}}$: always consistent, with dimension $4 - 2 = 2$.

(c) There is $n - \operatorname{rank} A = 4 - 3 = 1$ free variable. For each $\vec{\mathbf{b}}$: either $\operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,] = 4$, and there is **no solution**; or $\operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,] = 3$, and there are **infinitely many** (a line). **Never exactly one.**

(d) Yes, when $\operatorname{rank} A = \operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,] \le 2$. Example: $x + y + z = 1$, $2x + 2y + 2z = 2$, $3x + 3y + 3z = 3$, $4x + 4y + 4z = 4$: rank 1, a whole plane of solutions.

(e) If $\operatorname{rank} A = 3$, then $\operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,] = 3$ for every $\vec{\mathbf{b}}$ (3 rows) and the system would always be consistent. So $\operatorname{rank} A < 3$: there is at least one free variable. $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ is always consistent, so it has **infinitely many** solutions, in particular nonzero ones.

Exercise B5 COMPUTE ⋆ *The rank theorem*

Count the pivots among the columns of A (left of the bar) for $\operatorname{rank} A$, and in all the columns for $\operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,]$. If the ranks are equal, there are $n - \operatorname{rank} A$ free variables (the columns of A without a pivot) and the solution set has dimension $n - \operatorname{rank} A$.

	n	$\operatorname{rank} A$	$\operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,]$	consistent?	free variables	dim.	solutions
(a)	3	3	3	yes	none	0	exactly one (a point)
(b)	4	2	2	yes	x_2, x_4	2	infinitely many (a plane)
(c)	3	2	3	no	—	—	none
(d)	3	2	2	yes	z	1	infinitely many (a line)
(e)	2	2	2	yes	none	0	exactly one (a point)
(f)	3	1	1	yes	x, z	2	infinitely many (a plane)

- (a) The last row reads $5z = 0$: it is **not** a contradiction, just $z = 0$. Three pivots in the three columns of A.
- (b) Pivots in columns 1 and 3, so x_1, x_3 are pivot variables and x_2, x_4 are free: $4 - 2 = 2$ parameters.
- (c) The last row $[\,0\ 0\ 0 \mid 2\,]$ reads $0 = 2$: the extra pivot is in the bar column, $\operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,] = \operatorname{rank} A + 1$.
- (d) Two zero rows, but only one free variable: the dimension is $n - \operatorname{rank} A = 3 - 2 = 1$, not "the number of zero rows".
- (e) A zero row and still **exactly one** solution: $\operatorname{rank} A = 2 = n$.
- (f) The only pivot is in column 2 (the column of y), so x and z are free. The first unknown can be free.

Exercise B6 TRUE / FALSE ⋆⋆ *The rank theorem*

(a) **False.** Equal ranks only mean that the system is **consistent**. There is exactly one solution when, in addition, $\operatorname{rank} A = n$. Counterexample: $x + y = 1$ has $\operatorname{rank} A = \operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,] = 1$ and infinitely many solutions ($n - \operatorname{rank} A = 1$ free variable).

(b) **True.** The extra pivot is in the last column, so an echelon form contains a row $[\,0 \cdots 0 \mid c\,]$ with $c \neq 0$, that is $0 = c$: no solution.

(c) **True.** $\operatorname{rank} A \leq 3$ (at most one pivot per row, and there are 3 rows), so the number of free variables is $n - \operatorname{rank} A \geq 5 - 3 = 2$.

(d) **True.** A line has dimension 1. The system is consistent (it has solutions), so its dimension is $n - \operatorname{rank} A = 3 - \operatorname{rank} A = 1$, hence $\operatorname{rank} A = 2$.

(e) **True.** $[\,A \mid \vec{\mathbf{b}}\,]$ has 4 rows, so $4 = \operatorname{rank} A \leq \operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,] \leq 4$: the ranks are equal for every $\vec{\mathbf{b}}$ (consistent), and there are $n - \operatorname{rank} A = 4 - 4 = 0$ free variables: exactly one solution.

(f) **True.** The solution set of $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is a plane, so it has dimension $2 = n - \operatorname{rank} A$. The system $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has the same matrix A, is always consistent, so its solution set also has dimension $n - \operatorname{rank} A = 2$: a plane (through the origin). The dimension depends on A only, not on $\vec{\mathbf{b}}$.

Exercise B7 COMPUTE ⋆⋆ *Homogeneous and non-homogeneous*

(a)
$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{bmatrix} \mathbf{1} & 0 & -4 \\ 0 & \mathbf{1} & 3 \end{bmatrix}$$
$\operatorname{rank} A = 2$, $n = 3$: one free variable $z = t$; $x = 4t$, $y = -3t$:
$$\vec{\mathbf{h}} = t(4, -3, 1), \qquad t \in \mathbb{R}.$$
Geometrically: a **line through the origin** with direction vector $(4, -3, 1)$ (the intersection of two planes through O).

(b) $-2 + 3 - 0 = 1$ ✓ and $-4 + 9 + 0 = 5$ ✓: $\vec{\mathbf{p}}$ is a solution. The two systems have the same coefficients, so all the solutions are
$$\vec{\mathbf{x}} = \vec{\mathbf{p}} + \vec{\mathbf{h}} = (-2, 3, 0) + t(4, -3, 1), \qquad t \in \mathbb{R}.$$

(c) $(6, -3, 2) - (-2, 3, 0) = (8, -6, 2) = 2(4, -3, 1)$: it is the point $t = 2$ of the formula, so **yes**. Substitution: $6 - 3 - 2 = 1$ ✓ and $12 - 9 + 2 = 5$ ✓

Exercise B8 COMPUTE ⋆⋆ *Parameters*

(a) Pivots without parameter:
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & a & 2 \\ 2 & 3 & 4 & b \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & a-1 & 1 \\ 0 & 1 & 2 & b-2 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_2} \left[\begin{array}{ccc|c} \mathbf{1} & 1 & 1 & 1 \\ 0 & \mathbf{1} & a-1 & 1 \\ 0 & 0 & 3-a & b-3 \end{array}\right]$$

case	$\operatorname{rank} A$	$\operatorname{rank}[A \mid \vec{\mathbf{b}}]$	solutions
$a \neq 3$	3	3	exactly one
$a = 3,\ b \neq 3$	2	3	none (last row $0 = b - 3 \neq 0$)
$a = 3,\ b = 3$	2	2	infinitely many: a line

Case $a \neq 3$. $z = \dfrac{b-3}{3-a}$, then $y = 1 - (a-1)z$, and $x = 1 - y - z = (a-2)z$:
$$(x, y, z) = \Big(\frac{(a-2)(b-3)}{3-a},\ 1 - \frac{(a-1)(b-3)}{3-a},\ \frac{b-3}{3-a}\Big).$$
(Check with $a = 2$, $b = 4$: $(0, 0, 1)$ gives $1 = 1$, $2 = 2$, $4 = 4$ ✓)

Case $a = b = 3$. The last row is zero; $z = t$ is free, $y = 1 - 2t$, $x = 1 - y - z = t$:
$$(x, y, z) = (0, 1, 0) + t(1, -2, 1), \qquad t \in \mathbb{R}.$$
Check: $t + (1 - 2t) + t = 1$, $t + 2 - 4t + 3t = 2$, $2t + 3 - 6t + 4t = 3$ ✓

Common mistake

Dividing the last row by $3-a$ is allowed only when $a \neq 3$; the case $a=3$ is read from the matrix *before* any division.

(b) The system is homogeneous: it always has $\vec{\mathbf{0}}$, and it has a nonzero solution exactly when there is a free variable. Put the row without k in front first ($R_1 \leftrightarrow R_3$):

$$\begin{bmatrix} 1 & 1 & k \\ 1 & k & 1 \\ k & 1 & 1 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 - kR_1]{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 1 & 1 & k \\ 0 & k-1 & 1-k \\ 0 & 1-k & 1-k^2 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + R_2} \begin{bmatrix} 1 & 1 & k \\ 0 & k-1 & 1-k \\ 0 & 0 & 2-k-k^2 \end{bmatrix}$$

with $2-k-k^2 = -(k-1)(k+2)$.

- $k \neq 1$ and $k \neq -2$: three pivots ($k-1 \neq 0$ and $-(k-1)(k+2) \neq 0$), $\operatorname{rank} A = 3$: only $\vec{\mathbf{0}}$.
- $k = 1$: the matrix becomes $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\operatorname{rank} A = 1$, two free variables $y = s$, $z = t$:

$$(x, y, z) = s(-1, 1, 0) + t(-1, 0, 1) :$$

  the **plane** $x + y + z = 0$ through the origin (the three equations are the same).
- $k = -2$: the matrix becomes $\begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$, $\operatorname{rank} A = 2$, one free variable $z = t$: $y = t$, $x = -y + 2z = t$:

$$(x, y, z) = t(1, 1, 1) :$$

  a **line** through the origin. Check: each equation gives $-2t + t + t = 0$ ✓

Nonzero solutions exist exactly for $k = 1$ and $k = -2$.

Exercise B9 COMPUTE ⋆⋆ *The inverse with Gauss–Jordan*

(a)

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 3 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_2} \left[\begin{array}{ccc|ccc} \mathbf{1} & 2 & 1 & 1 & 0 & 0 \\ 0 & \mathbf{1} & 2 & 0 & 1 & 0 \\ 0 & 0 & \mathbf{1} & -1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[R_1 \leftarrow R_1 - R_3]{R_2 \leftarrow R_2 - 2R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & 2 & 3 & -2 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -5 & 3 \\ 0 & 1 & 0 & 2 & 3 & -2 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & -5 & 3 \\ 2 & 3 & -2 \\ -1 & -1 & 1 \end{bmatrix}.$$

Check two entries of AA^{-1}: row 1 · column 1 $= -2+4-1 = 1$ ✓; row 1 · column 2 $= -5+6-1 = 0$ ✓

(b) $\vec{\mathbf{x}} = A^{-1}\vec{\mathbf{b}}$:

$$A^{-1}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}-2-10+9\\2+6-6\\-1-2+3\end{bmatrix} = \begin{bmatrix}-3\\2\\0\end{bmatrix}, \qquad A^{-1}\begin{bmatrix}0\\1\\-1\end{bmatrix} = \begin{bmatrix}0-5-3\\0+3+2\\0-1-1\end{bmatrix} = \begin{bmatrix}-8\\5\\-2\end{bmatrix}.$$

Check: $A(-3,2,0) = (-3+4,\ 2,\ -3+6) = (1,2,3)$ ✓; $A(-8,5,-2) = (-8+10-2,\ 5-4,\ -8+15-8) = (0,1,-1)$ ✓

(c) $\operatorname{rank} A = 3$: the left part of $[\,A \mid I\,]$ was reduced to I_3, which has three pivots. (Equivalently: a 3×3 matrix is invertible if and only if its rank is 3.)

Exercise B10 COMPUTE ⋆ *Coffee sales*

Let s, m, ℓ be the numbers of small, medium and large coffees sold.

$$\begin{cases} s+m+\ell = 120 & \text{(number of coffees)}\\ 2s+3m+4\ell = 340 & \text{(revenue, in \$)}\\ s-m+\ell = 0 & (m = s+\ell)\end{cases}$$

$$\left[\begin{array}{ccc|c}1&1&1&120\\2&3&4&340\\1&-1&1&0\end{array}\right] \xrightarrow[R_3\leftarrow R_3-R_1]{R_2\leftarrow R_2-2R_1} \left[\begin{array}{ccc|c}1&1&1&120\\0&1&2&100\\0&-2&0&-120\end{array}\right] \xrightarrow{R_3\leftarrow R_3+2R_2} \left[\begin{array}{ccc|c}\mathbf{1}&1&1&120\\0&\mathbf{1}&2&100\\0&0&\mathbf{4}&80\end{array}\right]$$

$4\ell = 80 \Rightarrow \ell = 20$; $m = 100-40 = 60$; $s = 120-60-20 = 40$. **Answer:** 40 small, 60 medium and 20 large coffees.
Check in the statement: $40+60+20 = 120$ coffees ✓; $80+180+80 = 340$ dollars ✓; $60 = 40+20$ ✓

Exercise B11 COMPUTE ⋆⋆ *Balancing a chemical equation*

(a) Count each kind of atom on both sides:

$$\text{C: } 3a = c, \qquad \text{H: } 8a = 2d, \qquad \text{O: } 2b = 2c+d, \qquad \text{i.e.} \qquad \begin{cases} 3a-c = 0\\ 8a-2d = 0\\ 2b-2c-d = 0\end{cases}$$

A **homogeneous** system of 3 equations in 4 unknowns a, b, c, d.

(b) Coefficient matrix (columns a, b, c, d), with $R_2 \leftrightarrow R_3$, then $R_3 \leftarrow 3R_3 - 8R_1$ (scale by $3 \neq 0$, then replace: avoids fractions):

$$\begin{bmatrix}3&0&-1&0\\8&0&0&-2\\0&2&-2&-1\end{bmatrix} \xrightarrow{R_2\leftrightarrow R_3} \begin{bmatrix}3&0&-1&0\\0&2&-2&-1\\8&0&0&-2\end{bmatrix} \xrightarrow{R_3\leftarrow 3R_3-8R_1} \begin{bmatrix}\mathbf{3}&0&-1&0\\0&\mathbf{2}&-2&-1\\0&0&\mathbf{8}&-6\end{bmatrix}$$

Three pivots: rank $= 3$, so $4-3 = 1$ free variable, $d = t$. Back substitution: $8c = 6t \Rightarrow c = \frac{3}{4}t$; $2b = 2c+d = \frac{5}{2}t \Rightarrow b = \frac{5}{4}t$; $3a = c \Rightarrow a = \frac{1}{4}t$:

$$(a,b,c,d) = t\left(\tfrac{1}{4}, \tfrac{5}{4}, \tfrac{3}{4}, 1\right), \qquad t \in \mathbb{R}.$$

(c) $t = 4$ gives the smallest positive whole numbers: $(a, b, c, d) = (1, 5, 3, 4)$:

$$C_3H_8 + 5\,O_2 \longrightarrow 3\,CO_2 + 4\,H_2O.$$

Check: C $3 = 3$, H $8 = 8$, O $10 = 6+4$ ✓. The free variable means that the coefficients are only determined **up to a common factor**: multiplying all of them by the same number keeps the equation balanced.

Exercise B12 TRUE / FALSE ⋆⋆ *Systems*

(a) **False.** System B3(b) has 3 equations, 2 unknowns and the solution $(2, 1)$.

(b) **True.** $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has only $\vec{\mathbf{0}}$, so there is no free variable: $\operatorname{rank} A = 3 = n$. For a given $\vec{\mathbf{b}}$, either the system is inconsistent, or it is consistent with $n - \operatorname{rank} A = 0$ free variables, hence exactly one solution. In both cases: at most one.

(c) **False.** $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$: $A\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has infinitely many solutions (the line $x + y = 1$), but $A\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ has none ($x + y = 1$ and $x + y = 2$).

(d) **True.** Row operations do not change the solutions. Each system is therefore equivalent to the system of the common RREF, so both have the same solutions as that system.

(e) **False.** Swapping two columns swaps the roles of two unknowns. $x = 1$, $y = 2$ (that is $A = I_2$, $\vec{\mathbf{b}} = (1, 2)$) has the solution $(1, 2)$; after swapping the columns, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ gives $y = 1$, $x = 2$: the solution $(2, 1)$.

(f) **True.** If A has m rows and $\operatorname{rank} A = m$: $\operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,] \leq m$ (only m rows) and $\operatorname{rank}[\,A \mid \vec{\mathbf{b}}\,] \geq \operatorname{rank} A = m$. So the two ranks are equal for every $\vec{\mathbf{b}}$: always consistent.

Part C — Both chapters

Exercise C1 COMPUTE ⋆⋆ *A matrix equation $AX = B$*

(a) A is 3×3 and B is 3×2. For AX to be defined, X has 3 rows; for AX to be 3×2, it has 2 columns: X is 3×2.

(b) Write $X = [\,\vec{\mathbf{x}}_1 \;\; \vec{\mathbf{x}}_2\,]$ and $B = [\,\vec{\mathbf{b}}_1 \;\; \vec{\mathbf{b}}_2\,]$. Column j of AX is $A\vec{\mathbf{x}}_j$ (row i of A times column j of X, for each i). So

$$AX = B \iff A\vec{\mathbf{x}}_1 = \vec{\mathbf{b}}_1 \;\text{ and }\; A\vec{\mathbf{x}}_2 = \vec{\mathbf{b}}_2 :$$

two systems with the same coefficient matrix A. Reducing $[\,A \mid \vec{\mathbf{b}}_1 \;\vec{\mathbf{b}}_2\,]$ solves both at once (as $[\,A \mid I\,]$ does for A^{-1}).

(c)

$$\left[\begin{array}{ccc|cc} 1 & 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & 2 & 0 \\ 1 & 0 & 2 & 1 & 2 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|cc} 1 & 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & -1 & 2 & -2 & 3 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 + R_2} \left[\begin{array}{ccc|cc} \mathbf{1} & 1 & 0 & 3 & -1 \\ 0 & \mathbf{1} & 1 & 2 & 0 \\ 0 & 0 & \mathbf{3} & 0 & 3 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow \frac{1}{3} R_3} \left[\begin{array}{ccc|cc} 1 & 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - R_3} \left[\begin{array}{ccc|cc} 1 & 1 & 0 & 3 & -1 \\ 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

$$X = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}.$$

Check: $AX = \begin{bmatrix} 1+2+0 & 0-1+0 \\ 0+2+0 & 0-1+1 \\ 1+0+0 & 0+0+2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 0 \\ 1 & 2 \end{bmatrix} = B$ ✓

Exercise C2 TRUE / FALSE ⋆⋆ *Both chapters*

(a) **True.** $\operatorname{rank} A = 3$, so A is invertible. Then $A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$ and $(A^{-1})^T A^T = (AA^{-1})^T = I$: A^T is invertible, with $(A^T)^{-1} = (A^{-1})^T$.

(b) **True.** Multiply $AB = O$ on the left by A^{-1}: $B = IB = A^{-1}AB = A^{-1}O = O$.

(c) **True.** $\operatorname{rank} A \leq 2$ (only 2 rows) < 3 unknowns, so there is at least one free variable. A homogeneous system is always consistent, so it has infinitely many solutions; giving the free variable the value 1 gives a nonzero one.

(d) **False.** $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$: $P^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$, but $P \neq \pm I$. (Another one: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.)

(e) **False.** The third equation reads $0x + 0y + 0z = b_3$. With $\vec{\mathbf{b}} = (0, 0, 1)$ it becomes $0 = 1$: no solution. Here $\operatorname{rank} A = 2 < 3 = \operatorname{rank}[\, A \mid \vec{\mathbf{b}} \,]$.

(f) **True.** By (a), A^T is invertible; a product of invertible matrices is invertible: $(A^T A)^{-1} = A^{-1}(A^T)^{-1}$.