

Chapter 1 - What is a vector ?

From the Real Line to $\mathbb{R}^n$

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Today's Session

- 1 Why Linear Algebra?
- 2 The Real Line $\mathbb{R}$
- 3 The Plane $\mathbb{R}^2$
- 4 Space $\mathbb{R}^3$
- 5 Generalizing: $\mathbb{R}^n$
- 6 Points and Vectors: Two Different Roles
- 7 Scalar Multiplication
- 8 Addition and Subtraction
- 9 Vectors Between Points and Geometric Equalities

Why do scientists need linear algebra?

- You already know how to describe **one** quantity changing: a function $f(x)$, a single number evolving in time.
- But almost nothing in nature is described by a single number.

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Big idea

Linear algebra is the mathematics of **many numbers handled together, as one object.**

Nature Rarely Gives Us Just One Number

Physics

- Position of a particle: 3 numbers (x, y, z)
- Velocity: 3 more numbers
- Electric field at a point: 3 numbers, at every point in space

Chemistry

- Concentrations of n reacting species, all changing together

Biology

- Gene expression levels: thousands of numbers per cell

Data & Measurement

- Any experiment with n sensors produces one data point with n coordinates

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Each of these is a single point living in a space of many dimensions.

From Many Numbers to One Object

The key shift in viewpoint

Instead of treating $(x_1, x_2, \dots, x_n)$ as n separate numbers, we treat the whole list as **one mathematical object**: a **vector**.

This buys us something powerful: we can build rules — addition, scaling, later: products, transformations — that act on the *whole object at once*.

Example

A force in 3D space is not really “three numbers.” It is *one* arrow with a length and a direction. Linear algebra lets us add forces, scale them, and rotate them, all while treating each as a single thing.

Why This Matters: Examples You Already Know

GPS and maps

Your phone locates you with two numbers (latitude, longitude). Add altitude and you need three. Move to a city with 10,000 streets and every intersection has 2 coordinates. Linear algebra handles all of this in one framework.

Physics: forces

Push a box on a table. The push has a magnitude *and* a direction — two numbers in 2D, three in 3D. To find the net force from several pushes, you combine them as vectors.

Big idea

Our goal this session: build the space $\mathbb{R}^n$ where all of this lives, starting from something you already know — the real line.

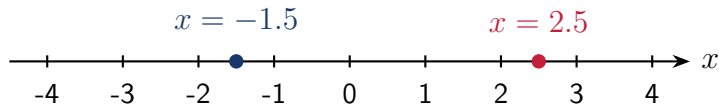
Starting Point: The Real Line

The set $\mathbb{R}$

$\mathbb{R}$ is the set of all real numbers. Geometrically, we picture it as a line: the **real line**.

- Every real number corresponds to exactly **one** point on the line.
- Every point on the line corresponds to exactly **one** real number.
- This is a **one-to-one correspondence** between numbers and positions.

The Real Line, Visually



- The number 0 is the **origin**: the reference point.
- A single coordinate x tells you *how far* and *in which direction* from the origin.
- One number is all you need to pin down any position on this line.

What We'll Carry Forward

Even though $\mathbb{R}$ looks simple, it already contains the seeds of everything we will build today:

- A choice of **origin** (reference point).
- A choice of **unit length** and **positive direction** (the scale).
- A way to name every position with **coordinates**.

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Question: what if one number isn't enough to describe where something is?

One Number Isn't Always Enough

- Describing a position on a map needs **two** numbers: how far East/West, how far North/South.
- Describing a point on a flat screen needs **two** numbers: horizontal and vertical position.

The plane $\mathbb{R}^2$

$\mathbb{R}^2$ is the set of all **ordered pairs** of real numbers:

$$\mathbb{R}^2 = \{(x, y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}.$$

Ordered matters: $(2, 5)$ and $(5, 2)$ are different points.

Building the Coordinate System in $\mathbb{R}^2$

To turn $\mathbb{R}^2$ into a picture, we make three choices:

- 1 An **origin**, labeled $O = (0, 0)$.
- 2 Two perpendicular number lines through O : the x -axis (horizontal) and the y -axis (vertical).
- 3 A common **unit length** on both axes.

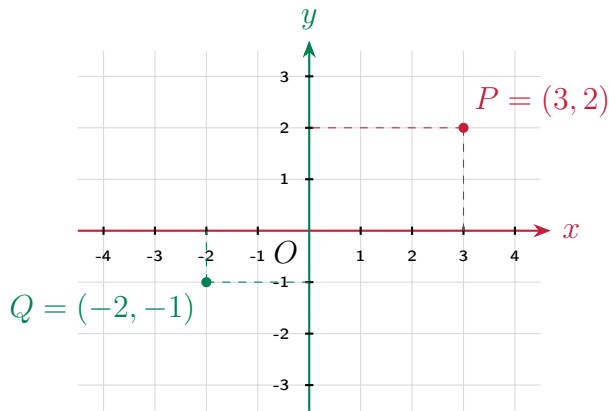
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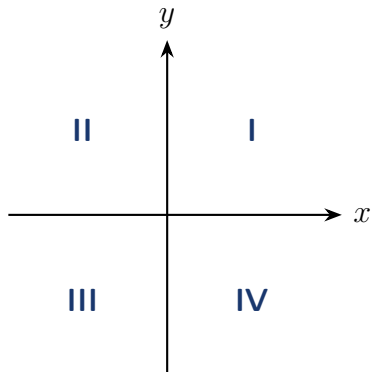
Each axis is just a copy of $\mathbb{R}$ — the number line from before — placed in a particular direction.

The Coordinate Plane



- The first coordinate (x) is read off the horizontal axis.
- The second coordinate (y) is read off the vertical axis.
- Dashed lines show how each coordinate is a **projection** onto an axis.

The Four Quadrants



Quadrant I: $x > 0, y > 0$

Quadrant II: $x < 0, y > 0$

Quadrant III: $x < 0, y < 0$

Quadrant IV: $x > 0, y < 0$

Points exactly on an axis have one coordinate equal to 0.

Quick Check

Without plotting, identify the quadrant of each point, or state if it lies on an axis:

Point	Sign of x	Sign of y
$(4, 7)$		
$(-3, 2)$		
$(0, -5)$		
$(-1, -6)$		

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Quadrant I | Quadrant II | y -axis | Quadrant III

Adding a Third Dimension

- Real space has **depth**, not just width and height.
- A point in a room needs three numbers: how far along the floor in two directions, and how high up.

Space $\mathbb{R}^3$

$\mathbb{R}^3$ is the set of all **ordered triples** of real numbers:

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}.$$

Same idea as $\mathbb{R}^2$, with one more axis: z .

Three Mutually Perpendicular Axes

We add a third number line, the z -axis, perpendicular to *both* the x -axis and the y -axis, through the same origin O .

The catch

Paper and screens are flat. To draw three mutually perpendicular axes on a 2D page, we must use **perspective**: one axis will look “slanted” even though all three are really perpendicular to each other.

Which Way Does z Point? The Right-Hand Rule

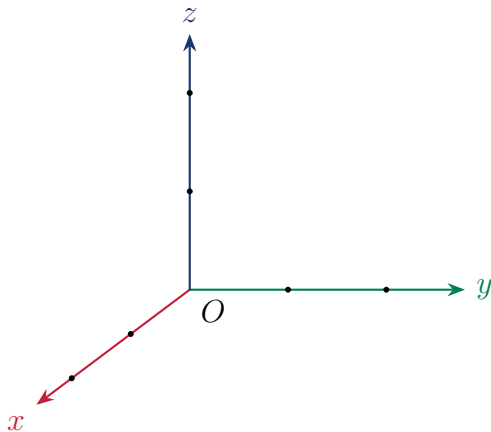
There are two ways to arrange x , y , z so they are mutually perpendicular: a left-handed way and a right-handed way. Mathematics and physics standardize on the **right-hand rule**.

Right-hand rule

Point the fingers of your right hand along the positive x -axis, then curl them toward the positive y -axis. Your thumb points along the positive z -axis.

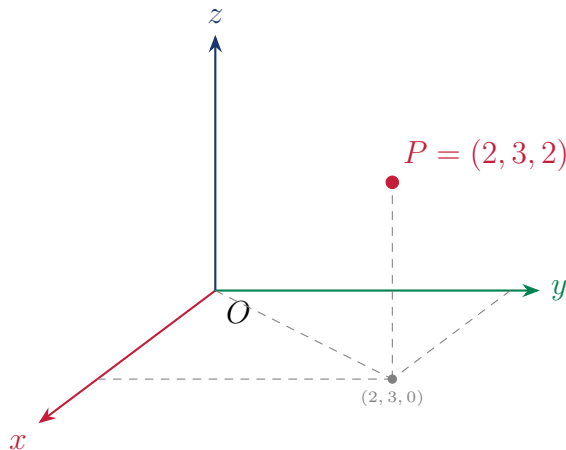
Equivalently: looking down from the positive z -axis toward the origin, the x -axis rotated 90° counterclockwise lands on the y -axis.

The Coordinate Axes in $\mathbb{R}^3$



Standard convention: x toward the viewer, y to the right, z upward. This arrangement is right-handed: fingers curl from x to y , thumb points along z .

Plotting a Point in $\mathbb{R}^3$

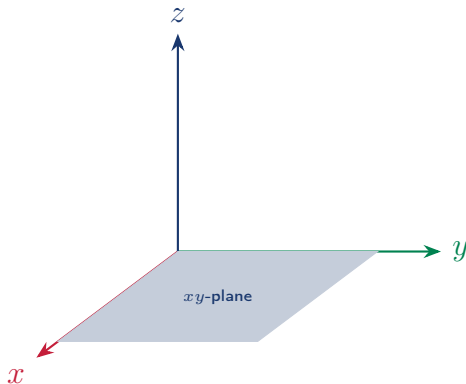


- First locate $(x, y) = (2, 3)$ in the *floor* plane, just like in $\mathbb{R}^2$.
- Then rise a height $z = 2$ straight up from that floor point.

The Coordinate Planes of $\mathbb{R}^3$

Each pair of axes spans a flat **coordinate plane**:

- the xy -plane: all points with $z = 0$,
- the xz -plane: all points with $y = 0$,
- the yz -plane: all points with $x = 0$.



The Octants

Just as $\mathbb{R}^2$ has four quadrants, $\mathbb{R}^3$ is divided by the three coordinate planes into **eight octants**, according to the signs of (x, y, z) .

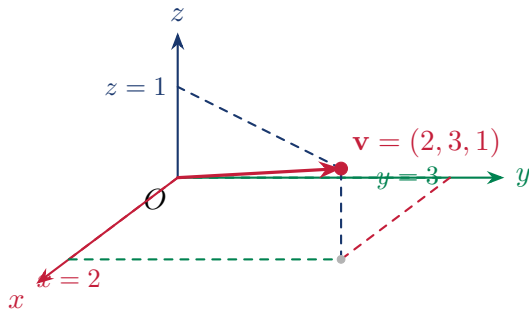
Example

The point $(2, 3, 2)$ has all positive coordinates, so it lies in the **first octant** — the natural analogue of Quadrant I.

We rarely name all eight individually; what matters is that the *sign pattern* of the coordinates tells you which region of space a point is in.

A Vector in $\mathbb{R}^3$

Every vector in $\mathbb{R}^3$ can be represented by an arrow from O . Its three components tell you how far to go along each axis.



Read the components off the axes:

- $x = 2$: 2 units along x
- $y = 3$: 3 units along y
- $z = 1$: 1 unit up

The Pattern So Far

Space	Elements	# of numbers needed
$\mathbb{R}$	x	1
$\mathbb{R}^2$	(x, y)	2
$\mathbb{R}^3$	(x, y, z)	3

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Space	Elements	# of numbers needed
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$\mathbb{R}^3$	(x, y, z)	3

Nothing in the algebra stops us at 3.

Defining $\mathbb{R}^n$

The space $\mathbb{R}^n$

For a positive integer n , $\mathbb{R}^n$ is the set of all **ordered n -tuples** of real numbers:

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_1, x_2, \dots, x_n \in \mathbb{R}\}.$$

- Each x_i is called a **component** (or **coordinate**) of the n -tuple.
- n is the **dimension** of the space.
- $\mathbb{R}^1 = \mathbb{R}$, the familiar number line.

Writing Vectors: Notation

A vector in $\mathbb{R}^n$ is written as an ordered list:

$$\mathbf{v} = (v_1, v_2, \dots, v_n), \quad \text{each } v_i \in \mathbb{R}.$$

- We use **bold** ($\mathbf{v}$) or an arrow ($\vec{v}$) to remind ourselves it is a vector, not a single number.
- Each v_i is called a **component** of $\mathbf{v}$.
- Two vectors are **equal** when every component matches:

$$(1, 2, 3) = (1, 2, 3) \quad \text{but} \quad (1, 2, 3) \neq (1, 3, 2).$$

- The **zero vector** $\mathbf{0} = (0, 0, \dots, 0)$ is the vector with all components 0.

*A note on “length”: for now we compare arrow lengths visually. The precise formula (the **norm**, $\|\mathbf{v}\| = \sqrt{v_1^2 + \dots + v_n^2}$) is coming in the next chapter.*

Can We Still Picture It?

$\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$:

- We can draw them.
- Geometric intuition (length, angle, direction) is built directly from the picture.

$\mathbb{R}^n$ for $n \geq 4$:

- We *cannot* draw it — our world only has 3 spatial dimensions.
- But the *algebra* (the rules for combining vectors) generalizes perfectly.

Big idea

Strategy: define every operation algebraically, component by component. Then use $\mathbb{R}^2$ and $\mathbb{R}^3$ as a **visual dictionary** to build intuition for what the algebra means, even in dimensions we cannot draw.

Why Higher Dimensions Show Up in Science

State of a physical system

To describe the motion of one particle in 3D space, you need its position *and* velocity: 6 numbers. For N particles (a gas, a molecule), that is $6N$ numbers — one point in $\mathbb{R}^{6N}$.

Data with several measurements

A medical record with n measurements (height, weight, blood pressure, heart rate, ...) is one point in $\mathbb{R}^n$. Comparing patients means comparing points in $\mathbb{R}^n$.

We will never draw $\mathbb{R}^{6N}$, but every operation we define today works there exactly as it works in $\mathbb{R}^2$.

Same Triple of Numbers, Two Different Jobs

Picture a car driving along a curved road in space.

- At every instant, the car has a **position**: where it is, (x, y, z) .
- At every instant, the car *also* has a **velocity**: how fast and in which direction it is moving right now, also a triple of numbers (v_x, v_y, v_z) .

Same Triple of Numbers, Two Different Jobs

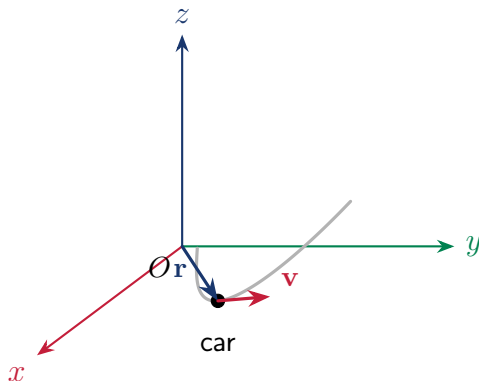
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Big idea

Both are triples in $\mathbb{R}^3$. Both are vectors. But they answer completely different questions, and we draw them differently.

Position vs. Velocity, Visually



- **Position $\mathbf{r}$:** drawn from the origin O . It tells you *where* the car is.
- **Velocity $\mathbf{v}$:** drawn from the *car itself*, tangent to the path. It tells you *how the position is changing*, not where anything is located.

Two Different Roles for a Vector

The car example shows us two distinct uses of the same kind of object:

Position vector $\mathbf{r}$

- Answers: *where is the car?*
- Always drawn **from the origin O** .
- Its components *are* the coordinates of the location.

Velocity vector $\mathbf{v}$

- Answers: *where is the car going?*
- Drawn **at the car** itself, tangent to the road.
- Only its *direction and length* matter — not where it starts.

Same algebra — different meaning

Both $\mathbf{r}$ and $\mathbf{v}$ are triples added and scaled by the same component-wise rules. The *mathematics* does not tell you which role a vector plays — the *context* does.

Why This Distinction Matters in Practice

- **Position** answers: *where is it?* Two objects with the same position are at the same place.
- **Velocity** (or force, acceleration, displacement, ...) answers: *what is happening to it right now?* Two objects can have the *same velocity* while being in completely different places.

Example

Two cars driving side by side on parallel straight roads can share the exact same velocity vector $\mathbf{v}$, even though their position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are completely different.

Big idea

From now on, whenever we draw a vector, ask whether it marks a *location* or describes a *change* — the algebra will not tell you which; the context will.

A First Operation: Scaling a Vector

- In $\mathbb{R}^2$ or $\mathbb{R}^3$, the most natural thing to do to an arrow is to **stretch**, **shrink**, or **reverse** it.
- This is **scalar multiplication**: multiplying a vector by a real number (a **scalar**).

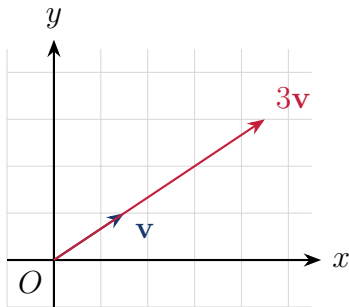
Scalar multiplication in $\mathbb{R}^n$

For a scalar $c \in \mathbb{R}$ and a vector $\mathbf{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$, define

$$c \mathbf{v} = (cv_1, cv_2, \dots, cv_n).$$

Each component is simply multiplied by c .

Scalar Multiplication in $\mathbb{R}^2$: Stretching

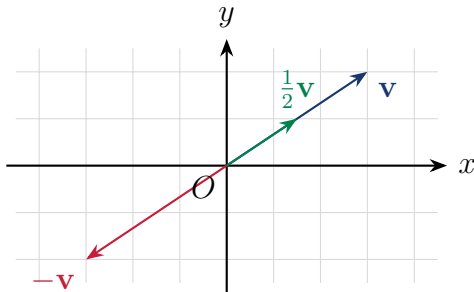


With $\mathbf{v} = (1.5, 1)$ and $c = 3$:

$$3\mathbf{v} = (3 \cdot 1.5, 3 \cdot 1) = (4.5, 3).$$

- Same direction as $\mathbf{v}$.
- Length is multiplied by 3.

Scalar Multiplication: Shrinking and Reversing

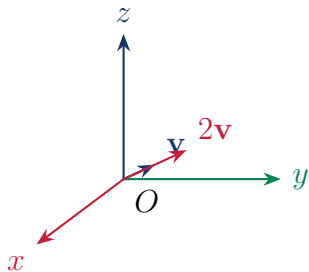


- $0 < c < 1$: same direction, **shorter** arrow ($\frac{1}{2}\mathbf{v}$ above).
- $c < 0$: **opposite** direction. In particular, $-\mathbf{v} = (-1)\mathbf{v}$ reverses $\mathbf{v}$ exactly.
- $c = 0$: $0\mathbf{v} = \mathbf{0}$, the zero vector, regardless of $\mathbf{v}$.

Scalar Multiplication in $\mathbb{R}^3$

The same rule, one more component:

$$c(v_1, v_2, v_3) = (cv_1, cv_2, cv_3).$$



Example

If $\mathbf{v} = (0.8, 1, 0.6)$, then $2\mathbf{v} = (1.6, 2, 1.2)$: same direction, twice the length.

Scalar Multiplication in $\mathbb{R}^n$: Same Idea

Big idea

For $\mathbf{v} \in \mathbb{R}^n$ with $n \geq 4$, we *cannot* draw an arrow getting longer or shorter. But algebraically, nothing changes:

$$c(v_1, v_2, \dots, v_n) = (cv_1, cv_2, \dots, cv_n).$$

Example

In $\mathbb{R}^4$: if $\mathbf{v} = (1, -2, 0, 3)$, then

$$-2\mathbf{v} = (-2)(1, -2, 0, 3) = (-2, 4, 0, -6).$$

We trust this is the “right” generalization precisely *because* it agrees with the geometric picture in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Quick Check: Scalar Multiplication

Let $\mathbf{u} = (2, -3, 0, 5)$ in $\mathbb{R}^4$.

- 1 Compute $4\mathbf{u}$.
- 2 Compute $-\mathbf{u}$.
- 3 Compute $0 \cdot \mathbf{u}$.

Quick Check: Scalar Multiplication

Let $\mathbf{u} = (2, -3, 0, 5)$ in $\mathbb{R}^4$.

- 1 Compute $4\mathbf{u}$.
- 2 Compute $-\mathbf{u}$.
- 3 Compute $0 \cdot \mathbf{u}$.

Answers

- 1 $4\mathbf{u} = (8, -12, 0, 20)$
- 2 $-\mathbf{u} = (-2, 3, 0, -5)$
- 3 $0 \cdot \mathbf{u} = (0, 0, 0, 0) = \mathbf{0}$

A Second Operation: Combining Vectors

- Scalar multiplication changes the *size* of a single vector.
- To combine *two* vectors into one, we need **addition**.

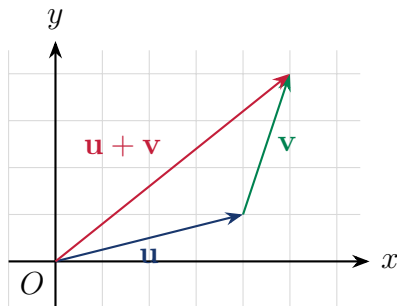
Addition in $\mathbb{R}^n$

For $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$ in $\mathbb{R}^n$,

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n).$$

Components are added pairwise — nothing more exotic than that.

Addition in $\mathbb{R}^2$: Tip-to-Tail



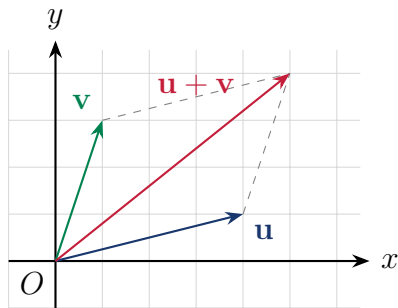
Tip-to-tail rule:

- Draw $\mathbf{u}$ from the origin.
- Draw $\mathbf{v}$ starting at the *tip* of $\mathbf{u}$ (not from O).
- The sum $\mathbf{u} + \mathbf{v}$ is the arrow from O to the tip of $\mathbf{v}$.

With $\mathbf{u} = (4, 1)$, $\mathbf{v} = (1, 3)$:

$$\mathbf{u} + \mathbf{v} = (5, 4).$$

Addition in $\mathbb{R}^2$: The Parallelogram Rule

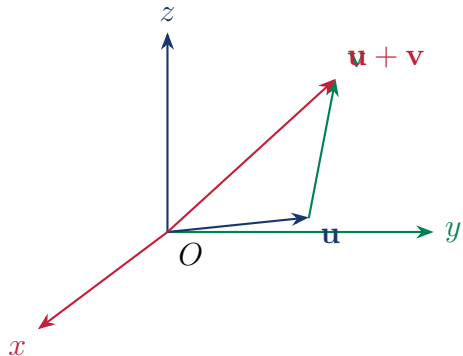


Equivalent picture: draw both $\mathbf{u}$ and $\mathbf{v}$ from O , complete the **parallelogram** they form.

- $\mathbf{u} + \mathbf{v}$ is the **diagonal** of the parallelogram, starting at O .
- Same answer as tip-to-tail — just a different way to see it.

Addition in $\mathbb{R}^3$

Tip-to-tail and the parallelogram rule work *exactly* the same way, one dimension up.



$$\mathbf{u} = (0.5, 2.0, 0.4), \quad \mathbf{v} = (0.3, 0.5, 1.8) \implies \mathbf{u} + \mathbf{v} = (0.8, 2.5, 2.2).$$

Subtraction: What Does $\mathbf{u} - \mathbf{v}$ Mean?

Subtraction in $\mathbb{R}^n$

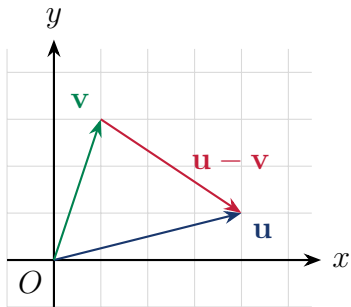
Subtraction is addition of the negative:

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = (u_1 - v_1, u_2 - v_2, \dots, u_n - v_n).$$

Big idea

Geometrically: $\mathbf{u} - \mathbf{v}$ is the arrow that goes **from the tip of $\mathbf{v}$ to the tip of $\mathbf{u}$** , when both are drawn from the same origin.

Subtraction in $\mathbb{R}^2$, Visually



With $\mathbf{u} = (4, 1)$, $\mathbf{v} = (1, 3)$:

$$\mathbf{u} - \mathbf{v} = (3, -2).$$

The arrow points from the tip of $\mathbf{v}$ toward the tip of $\mathbf{u}$ — it measures “how to get from $\mathbf{v}$ to $\mathbf{u}$.”

Quick Check: Addition and Subtraction

Let $\mathbf{u} = (3, -1, 2)$ and $\mathbf{v} = (-1, 4, 1)$ in $\mathbb{R}^3$.

- 1 Compute $\mathbf{u} + \mathbf{v}$.
- 2 Compute $\mathbf{u} - \mathbf{v}$.

Quick Check: Addition and Subtraction

Let $\mathbf{u} = (3, -1, 2)$ and $\mathbf{v} = (-1, 4, 1)$ in $\mathbb{R}^3$.

- 1 Compute $\mathbf{u} + \mathbf{v}$.
- 2 Compute $\mathbf{u} - \mathbf{v}$.

Answers

- 1 $\mathbf{u} + \mathbf{v} = (3 + (-1), -1 + 4, 2 + 1) = (2, 3, 3)$
- 2 $\mathbf{u} - \mathbf{v} = (3 - (-1), -1 - 4, 2 - 1) = (4, -5, 1)$

Notice: each result is still in $\mathbb{R}^3$, still a triple. Adding or subtracting vectors never changes the dimension.

A Key Property: Distributivity

Recall from real numbers: $c(a + b) = ca + cb$. The same law holds for vectors.

Numeric check first

Let $c = 3$, $\mathbf{u} = (1, 2)$, $\mathbf{v} = (3, -1)$.

$$\mathbf{u} + \mathbf{v} = (4, 1)$$

$$3(\mathbf{u} + \mathbf{v}) = (12, 3)$$

$$3\mathbf{u} + 3\mathbf{v} = (3, 6) + (9, -3) = (12, 3) \quad \checkmark$$

Distributivity of scalar multiplication

For any scalar $c \in \mathbb{R}$ and vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$:

$$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}, \quad c(\mathbf{u} - \mathbf{v}) = c\mathbf{u} - c\mathbf{v}.$$

Why Distributivity Holds: Component by Component

Write $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$. Then

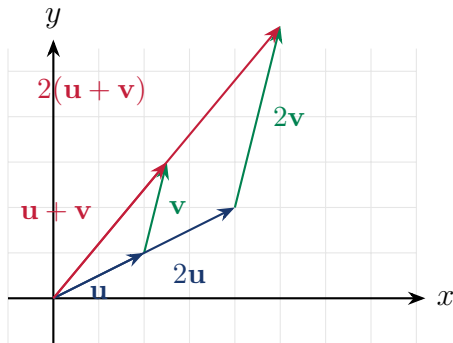
$$\begin{aligned} c(\mathbf{u} + \mathbf{v}) &= c(u_1 + v_1, \dots, u_n + v_n) \\ &= (c(u_1 + v_1), \dots, c(u_n + v_n)) \\ &= (cu_1 + cv_1, \dots, cu_n + cv_n) \quad (\text{real number distributivity, each component}) \\ &= (cu_1, \dots, cu_n) + (cv_1, \dots, cv_n) \\ &= c\mathbf{u} + c\mathbf{v}. \end{aligned}$$

Big idea

There is nothing new here: this is the ordinary distributive law of real numbers, applied to each component separately.

Distributivity: Seen Geometrically

Scaling the whole tip-to-tail picture by c gives the same result as adding the already-scaled vectors — this is $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$, seen with your eyes:



The small arrows ($\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} + \mathbf{v}$) and the doubled arrows ($2\mathbf{u}$, $2\mathbf{v}$, $2(\mathbf{u} + \mathbf{v})$) are the *same* tip-to-tail picture, just scaled by 2. The diagonal reaches the exact same endpoint either way.

Properties of Addition

For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, the following hold — all follow from the same laws for real numbers, applied component by component:

- **Commutativity:** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- **Associativity:** $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- **Zero vector:** $\mathbf{u} + \mathbf{0} = \mathbf{u}$
- **Additive inverse:** $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

Quick check: commutativity

$$(1, 3) + (-2, 5) = (-1, 8) \quad \text{and} \quad (-2, 5) + (1, 3) = (-1, 8). \quad \checkmark$$

Worked Example: Putting It Together

Let $\mathbf{u} = (2, -1, 3)$ and $\mathbf{v} = (-1, 4, 0)$ in $\mathbb{R}^3$. Compute $2\mathbf{u} - 3\mathbf{v}$.

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Solution

$$2\mathbf{u} = 2(2, -1, 3) = (4, -2, 6)$$

$$3\mathbf{v} = 3(-1, 4, 0) = (-3, 12, 0)$$

$$\begin{aligned} 2\mathbf{u} - 3\mathbf{v} &= (4, -2, 6) - (-3, 12, 0) \\ &= (4 - (-3), -2 - 12, 6 - 0) \\ &= (7, -14, 6) \end{aligned}$$

Every step is component-by-component; the same calculation works verbatim in $\mathbb{R}^n$ for any n .

From Two Points to a Vector

So far we have described a vector as an arrow *from the origin O* .

But in practice, we often know two **locations** A and B and want the vector that goes from one to the other.

Examples

- Two cities on a map: what is the displacement from city A to city B ?
- A robot at position A : what move takes it to target B ?

Big idea

We write $\overrightarrow{AB}$ for the vector that, **when applied at A , takes you to B** . It is not tied to A as a starting point — it is a displacement whose components are determined by A and B .

Formula: $\overrightarrow{AB} = B - A$

The vector $\overrightarrow{AB}$

$\overrightarrow{AB}$ is the vector that, applied at A , takes you to B . If $A = (a_1, \dots, a_n)$ and $B = (b_1, \dots, b_n)$, its components are:

$$\overrightarrow{AB} = B - A = (b_1 - a_1, b_2 - a_2, \dots, b_n - a_n).$$

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Example in $\mathbb{R}^2$

With $A = (1, 1)$ and $B = (4, 3)$: $\overrightarrow{AB} = (4 - 1, 3 - 1) = (3, 2)$.

Formula: $\overrightarrow{AB} = B - A$

The vector $\overrightarrow{AB}$

$\overrightarrow{AB}$ is the vector that, applied at A , takes you to B . If $A = (a_1, \dots, a_n)$ and $B = (b_1, \dots, b_n)$, its components are:

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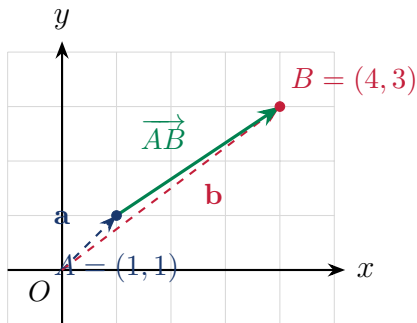
Example in $\mathbb{R}^2$

With $A = (1, 1)$ and $B = (4, 3)$: $\overrightarrow{AB} = (4 - 1, 3 - 1) = (3, 2)$.

Example in $\mathbb{R}^3$

With $A = (2, 0, -1)$ and $B = (5, 3, 2)$: $\overrightarrow{AB} = (5 - 2, 3 - 0, 2 - (-1)) = (3, 3, 3)$.

Why $B - A$? The Geometry



From the tip-to-tail rule:

$$\mathbf{a} + \overrightarrow{AB} = \mathbf{b}$$

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = B - A.$$

$\overrightarrow{AB}$ is the vector that, applied at A ,

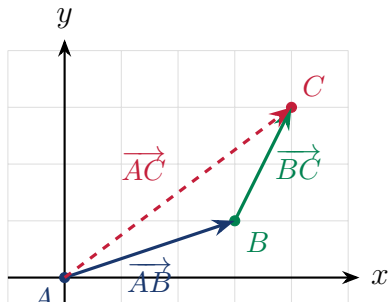
reaches B : its components are tip minus tail, i.e. $B - A$.

The Chasles Relation

Chasles relation

For any points A, B, C in $\mathbb{R}^n$: $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

Reading: going from A to C in one step, or via an intermediate point B , gives the same displacement vector.



The Chasles Relation: Verification and Consequences

Verification

With $A = (0, 0)$, $B = (3, 1)$, $C = (4, 3)$:

$$\overrightarrow{AB} + \overrightarrow{BC} = (3, 1) + (1, 2) = (4, 3) = \overrightarrow{AC} \quad \checkmark$$

Immediate consequences

- $\overrightarrow{AA} = \mathbf{0}$ (set $B = C = A$)
- $\overrightarrow{BA} = -\overrightarrow{AB}$ (since $\overrightarrow{AB} + \overrightarrow{BA} = \overrightarrow{AA} = \mathbf{0}$)
- Chaining: $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$

Equal, Collinear, and Parallel Vectors

Equal vectors

Two vectors are **equal** iff they are **parallel**, have the **same length**, and the **same orientation** (not pointing in opposite directions).

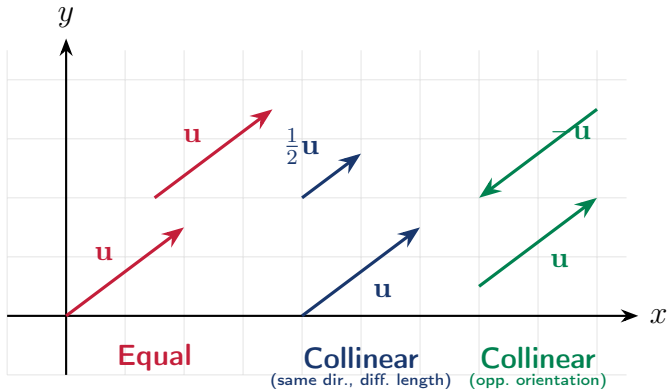
Collinear vectors

Two vectors are **collinear** if they are **parallel** — regardless of their length or orientation. They lie along the same line or parallel lines, but are *not* necessarily equal.

Key distinction

Parallel only tells you about direction. To be **equal**, length and orientation must also match. Two vectors pointing in **opposite** directions are collinear but **not** equal.

Visualizing Equality and Collinearity



In Components

For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$:

- **Equal:** $\mathbf{u} = \mathbf{v}$ iff $u_i = v_i$ for all i . All components match.
- **Collinear:** $\mathbf{u}$ and $\mathbf{v}$ are collinear iff $\mathbf{v} = \lambda \mathbf{u}$ for some scalar $\lambda \in \mathbb{R}$.
 - $\lambda > 0$: same orientation.
 - $\lambda < 0$: opposite orientation.
 - $\lambda = 1$: equal vectors.

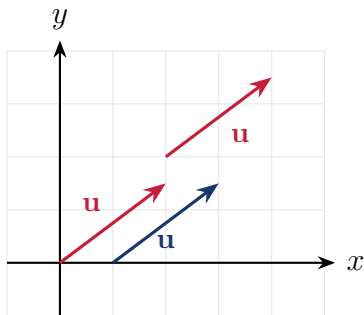
Examples in $\mathbb{R}^2$

$\mathbf{u} = (2, 1)$, $\mathbf{v} = (4, 2)$, $\mathbf{w} = (-2, -1)$.

$\mathbf{v} = 2\mathbf{u}$: collinear with $\mathbf{u}$, same orientation, *not* equal.

$\mathbf{w} = -\mathbf{u}$: collinear with $\mathbf{u}$, opposite orientation, *not* equal.

Equal Vectors in Practice: Three Arrows, One Vector



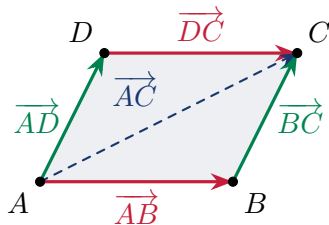
All three arrows represent the *same* vector $\mathbf{u} = (2, 1.5)$:

- Same length ✓
- Parallel (same line direction) ✓
- Same orientation (same sense) ✓

Where the arrow is drawn is irrelevant.
Only the **components** matter.

Equal Vectors on a Parallelogram

Let $ABCD$ be a parallelogram ($AB \parallel DC$, $AD \parallel BC$).



Since opposite sides are parallel and equal in length:

$$\overrightarrow{AB} = \overrightarrow{DC}$$

$$\overrightarrow{AD} = \overrightarrow{BC}$$

The diagonal equals the sum of two adjacent sides:

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD}$$

(The parallelogram rule from the addition section.)

Parallelogram Equalities: Algebraic Check

Verify with coordinates

Let $A = (0, 0)$, $B = (3, 0)$, $C = (4, 2)$, $D = (1, 2)$.

$$\overrightarrow{AB} = (3, 0) - (0, 0) = (3, 0)$$

$$\overrightarrow{DC} = (4, 2) - (1, 2) = (3, 0) \quad \checkmark$$

$$\overrightarrow{AD} = (1, 2) - (0, 0) = (1, 2)$$

$$\overrightarrow{BC} = (4, 2) - (3, 0) = (1, 2) \quad \checkmark$$

$$\overrightarrow{AC} = (4, 2) - (0, 0) = (4, 2)$$

$$\overrightarrow{AB} + \overrightarrow{AD} = (3, 0) + (1, 2) = (4, 2) \quad \checkmark$$

Quick Check: Is It a Parallelogram?

Given $A = (1, 2)$, $B = (4, 3)$, $C = (5, 6)$, $D = (2, 5)$,
verify that $ABCD$ is a parallelogram by checking $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$.

Quick Check: Is It a Parallelogram?

Given $A = (1, 2)$, $B = (4, 3)$, $C = (5, 6)$, $D = (2, 5)$,
verify that $ABCD$ is a parallelogram by checking $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$.

Solution

$$\overrightarrow{AB} = (4 - 1, 3 - 2) = (3, 1)$$

$$\overrightarrow{DC} = (5 - 2, 6 - 5) = (3, 1) \quad \checkmark$$

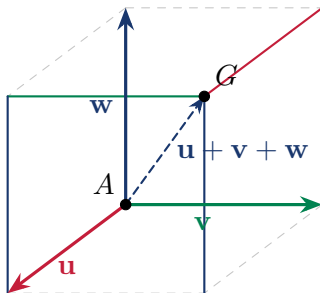
$$\overrightarrow{AD} = (2 - 1, 5 - 2) = (1, 3)$$

$$\overrightarrow{BC} = (5 - 4, 6 - 3) = (1, 3) \quad \checkmark$$

Both pairs of opposite sides are equal $\Rightarrow ABCD$ is a parallelogram.

Equal Vectors on a Parallelepiped

A **parallelepiped** is the 3D analogue of a parallelogram: a box with 3 pairs of parallel faces, built from three edge vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.



Parallelepiped: Equal Edges and Main Diagonal

The 12 edges of a parallelepiped group into **3 sets of 4 equal vectors**:

- 4 edges parallel to $\mathbf{u}$ (all equal to $\mathbf{u}$)
- 4 edges parallel to $\mathbf{v}$ (all equal to $\mathbf{v}$)
- 4 edges parallel to $\mathbf{w}$ (all equal to $\mathbf{w}$)

The **main diagonal** from A to the opposite vertex G equals the sum of all three edge vectors:

$$\overrightarrow{AG} = \mathbf{u} + \mathbf{v} + \mathbf{w}.$$

Big idea

This is the 3D generalization of the parallelogram diagonal rule $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD}$.

Summary (1/2)

- Linear algebra describes **many numbers as one object**: essential whenever a natural system needs more than one quantity to describe its state.
- $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$ are built from the same idea: a choice of origin, axes, and unit length, generalized one dimension at a time.
- $\mathbb{R}^n$ extends this to n -tuples $(x_1, \dots, x_n)$ — we lose the picture beyond $n = 3$, but keep all the algebra.
- A vector can mark a **location** (anchored at O) or describe a **rate or change** (like velocity on a moving car) — same algebra, different role.
- **Scalar multiplication** $c\mathbf{v}$ scales (and possibly reverses) a vector, component by component.
- **Addition/subtraction** combine vectors component by component; geometrically, tip-to-tail or the parallelogram rule.

Summary (2/2)

- $\overrightarrow{AB}$ is the vector that, **applied at A, takes you to B**: its components are $B - A$. It is not tied to A as a starting point.
- **Chasles relation**: $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ — going via an intermediate point gives the same displacement. Consequences: $\overrightarrow{AA} = \mathbf{0}$, $\overrightarrow{BA} = -\overrightarrow{AB}$.
- Two vectors are **equal** iff they are parallel, same length, and same orientation. They are **collinear** if only parallel (regardless of length or orientation).
- On a **parallelogram** $ABCD$: $\overrightarrow{AB} = \overrightarrow{DC}$, $\overrightarrow{AD} = \overrightarrow{BC}$, diagonal $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD}$.
- On a **parallelepiped**: 12 edges split into 3 groups of 4 equal vectors; main diagonal = $\mathbf{u} + \mathbf{v} + \mathbf{w}$.