Chapter 4 - Planes in Space

Definitions, Projection, Distance, and Intersections

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Today's Session

Learning goals

By the end of this chapter, you should be able to:

1. build a plane equation from points or a point and a normal;
2. interpret a normal vector and check whether a point lies in a plane;
3. find a projection and a distance;
4. classify and calculate line–plane and plane–plane intersections.

Before calculating, sketch or predict the geometry. After calculating, substitute to check your result.

Three Equivalent Ways to Pin Down a Plane

What determines a plane?

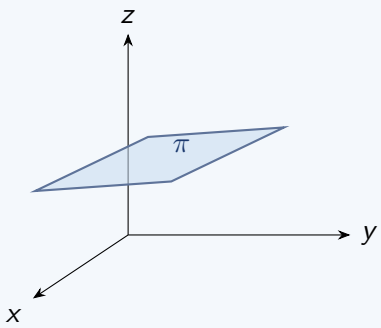
A plane in $\mathbb{R}^3$ is uniquely determined by any of the following:

- three **non-collinear points**,
- one **point** and a **nonzero normal vector**, or
- a **unit normal** oriented from the origin toward the plane, together with its **distance** p from the origin.

A plane needs two independent directions within it. Their cross product provides one direction perpendicular to the plane: a **normal**. A normal alone fixes orientation, but not position.

Visualizing a Plane in Space

Visualization



A plane extends infinitely in two independent directions; the parallelogram shown is just a convenient finite patch of it.

Definition: Plane Through Three Points

Three non-collinear points

Let A, B, C be non-collinear. With $\vec{u} = \overrightarrow{AB}$ and $\vec{v} = \overrightarrow{AC}$, every point P in their plane satisfies

$$\boxed{\overrightarrow{OP} = \overrightarrow{OA} + s\vec{u} + t\vec{v}, \qquad s, t \in \mathbb{R}.}$$

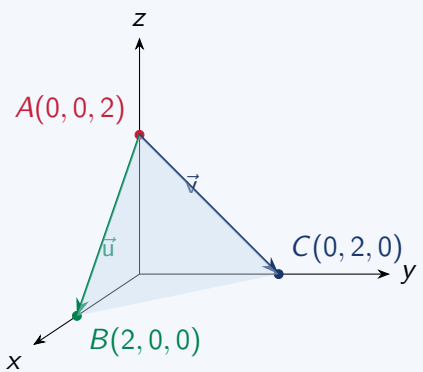
Equivalently, in coordinates, $(x, y, z) = A + s(B - A) + t(C - A)$.

Why non-collinear?

If $\vec{u}$ and $\vec{v}$ are parallel, varying s, t traces a line (or a point), and infinitely many planes can contain it.

From Three Points to Two Direction Vectors

Visualization



Given three non-collinear points A, B, C:

$$\vec{u} = \overrightarrow{AB} = B - A, \quad \vec{v} = \overrightarrow{AC} = C - A$$

are two **direction vectors** of the plane.

Example

$A(0, 0, 2)$, $B(2, 0, 0)$, $C(0, 2, 0)$:

$$\vec{u} = (2, 0, -2),$$
$$\vec{v} = (0, 2, -2).$$

From Two Directions to a Normal Vector

Big idea

The **cross product** $\vec{n} = \vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$, hence perpendicular to *every* vector lying in the plane they span. So $\vec{n}$ is a **normal vector** to the plane.

Example

$\vec{u} = (2, 0, -2)$, $\vec{v} = (0, 2, -2)$:

$$\vec{n} = \vec{u} \times \vec{v} = \big(0(-2) - (-2)(2),\ (-2)(0) - 2(-2),\ 2(2) - 0(0)\big) = (4, 4, 4)$$

Any nonzero scalar multiple works too, e.g. the simpler $\vec{n} = (1, 1, 1)$.

Definition: Plane Through a Point with a Given Normal

Plane determined by a point and a normal vector

Let P_0 be a point and $\vec{n} \neq \vec{0}$ a normal vector. The plane π through P_0 with normal $\vec{n}$ is the set of points P such that

$$\vec{n} \cdot \overrightarrow{P_0P} = 0.$$

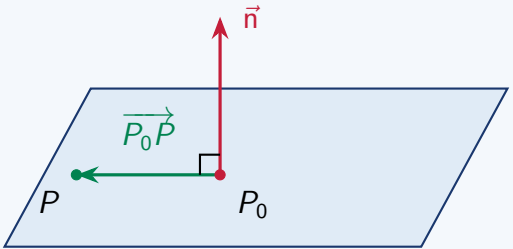
Writing $P = (x, y, z)$, $P_0 = (x_0, y_0, z_0)$, $\vec{n} = (A, B, C)$:

$$\boxed{A(x - x_0) + B(y - y_0) + C(z - z_0) = 0}$$

Expanding gives the **Cartesian equation** $Ax + By + Cz + D = 0$ with $D = -(Ax_0 + By_0 + Cz_0)$.

Visualizing the Point-Normal Definition

Visualization



Since P_0 and P lie in the plane, $\overrightarrow{P_0P}$ is a direction contained in the plane. The normal vector $\vec{n}$ is perpendicular to every such direction:

$$\boxed{\vec{n} \cdot \overrightarrow{P_0P} = 0}.$$

Example: Equation from a Point and a Normal

Find the Cartesian equation of the plane through $P_0(1, 1, 1)$ with normal $\vec{n} = (1, 1, 1)$.

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Find the Cartesian equation of the plane through $P_0(1,1,1)$ with normal $\vec{n} = (1,1,1)$.

1. $\vec{n} \cdot \overrightarrow{P_0P} = 0$: $\quad 1(x-1) + 1(y-1) + 1(z-1) = 0$.
2. Expand: $x + y + z - 3 = 0$.

Reading off the normal directly

Conversely, from $Ax + By + Cz + D = 0$ you can **read the normal instantly**: $\vec{n} = (A, B, C)$. Here $\vec{n} = (1,1,1)$, matching what we started with.

Quick Check: Points A, B, C to Cartesian Equation

Find the Cartesian equation of the plane through $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$.

Quick Check: Points A, B, C to Cartesian Equation

Find the Cartesian equation of the plane through $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$.

1. $\vec{u} = B - A = (-1,1,0), \quad \vec{v} = C - A = (-1,0,1)$.
2. $\vec{n} = \vec{u} \times \vec{v} = (1\cdot 1 - 0\cdot 0,\ 0\cdot(-1) - (-1)\cdot 1,\ (-1)\cdot 0 - 1\cdot(-1)) = (1,1,1)$.
3. Using $P_0 = A(1,0,0)$: $1(x-1) + 1(y-0) + 1(z-0) = 0 \implies x + y + z - 1 = 0$.

Check

Plug in $B(0,1,0)$: $0 + 1 + 0 - 1 = 0$ ✓. Plug in $C(0,0,1)$: $0 + 0 + 1 - 1 = 0$ ✓.

Definition: Plane Determined by a Normal Direction and a Distance

Normal form (Hesse form)

Choose a unit normal $\hat{\vec{n}}$ pointing *from the origin toward the plane*, and let $p = d(O, \pi) \geq 0$. Then

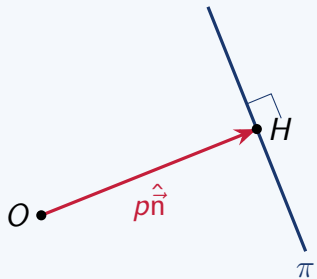
$$\boxed{\hat{\vec{n}} \cdot \overrightarrow{OP} = p \quad \text{for every } P \in \pi.}$$

For a plane through O, $p = 0$ and either orientation of the unit normal works. Otherwise, the orientation toward the plane is unique.

An arbitrary unit normal gives a signed right-hand side; its absolute value is the distance.

Visualizing the Normal-Distance Definition

Visualization



Edge-on cross section of the plane: $H = p\hat{\vec{n}}$ is its closest point to O. The normal displacement has length p.

Deriving the Distance p from the Cartesian Equation

From $Ax + By + Cz + D = 0$ to normal form

Start from $Ax + By + Cz + D = 0$, i.e. $\vec{n} \cdot (x, y, z) = -D$ with $\vec{n} = (A, B, C)$. Divide by $\|\vec{n}\|$ to make the normal a unit vector:

$$\hat{\vec{n}} \cdot (x, y, z) = \frac{-D}{\|\vec{n}\|}, \qquad \hat{\vec{n}} = \frac{(A, B, C)}{\|\vec{n}\|}$$

Choosing the sign of $\hat{\vec{n}}$ so the right-hand side is ≥ 0 gives the distance from the origin:

$$\boxed{p = \frac{|D|}{\sqrt{A^2 + B^2 + C^2}}}$$

Example: Normal Form and Distance from the Origin

Plane: $2x - 2y + z - 6 = 0$.

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1. $\vec{n} = (2, -2, 1)$, $\|\vec{n}\| = \sqrt{4 + 4 + 1} = 3$.
2. Normal form: $\dfrac{2x - 2y + z}{3} = 2$, so $\hat{\vec{n}} = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$.
3. Distance from the origin: $p = \dfrac{|-6|}{3} = 2$.

Sanity check

$D = -6 < 0$ and the right-hand side $2 > 0$ automatically, so no sign flip was needed here; if D had been positive we would flip the sign of both $\vec{n}$ and D before dividing.

Three Faces of the Same Plane

Big idea — three descriptions of one plane

Vector (parametric) form:	$(x, y, z) = A + s\vec{u} + t\vec{v}$
Point-normal form:	$\vec{n} \cdot \overrightarrow{P_0P} = 0$, i.e. $Ax + By + Cz + D = 0$
Normal (Hesse) form:	$\hat{\vec{n}} \cdot \overrightarrow{OP} = p$, i.e. $\hat{\vec{n}} \cdot (x, y, z) = p$

Every plane has all three; which one is most convenient depends on what you are given and what you want to compute.

Roadmap: Converting Between Definitions

Given	Procedure
A, B, C non-collinear	$\vec{u} = B - A$, $\vec{v} = C - A$; compute $\vec{n} = \vec{u} \times \vec{v}$; use A and $\vec{n}$.
Point P_0, normal $\vec{n}$	Form $\vec{n} \cdot \overrightarrow{P_0P} = 0$. For parametric form, find two independent solutions of $\vec{n} \cdot \vec{u} = 0$.
$Ax + By + Cz + D = 0$	Read $\vec{n} = (A, B, C)$; find one point by solving the equation; choose the sign and divide by $\|\vec{n}\|$ for Hesse form.
Unit normal $\hat{\vec{n}}$, p	Expand $\hat{\vec{n}} \cdot \overrightarrow{OP} = p$.

Always require $\vec{n} \neq \vec{0}$ and two *independent* direction vectors.

Example: From Point-Normal Back to Two Direction Vectors

Plane: $x + y + z - 3 = 0$, so $\vec{n} = (1, 1, 1)$, $P_0 = (1, 1, 1)$ on the plane (check: $1 + 1 + 1 - 3 = 0$ ✓).

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Plane: $x + y + z - 3 = 0$, so $\vec{n} = (1, 1, 1)$, $P_0 = (1, 1, 1)$ on the plane (check: $1 + 1 + 1 - 3 = 0$ ✓).

Finding $\vec{u}, \vec{v}$

We need two independent vectors $\perp \vec{n}$. Any vector (a, b, c) with $a + b + c = 0$ works; just pick two independent solutions.

Example

$$\vec{u} = (1, -1, 0) \quad (1 - 1 + 0 = 0 \checkmark), \qquad \vec{v} = (1, 0, -1) \quad (1 + 0 - 1 = 0 \checkmark)$$

$$(x, y, z) = (1, 1, 1) + s(1, -1, 0) + t(1, 0, -1)$$

Exercise: Round-Trip Between Definitions

Try first; predict whether D lies on the plane

The plane π passes through $A(2,0,0)$, $B(0,3,0)$, $C(0,0,6)$.

1. Find a normal and a Cartesian equation of π.
2. Give its distance from the origin and a unit normal pointing toward π.
3. Decide by substitution whether $D(1,1,1)$ lies on π.

Exercise: Round-Trip Between Definitions

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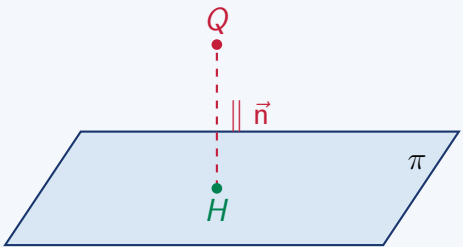
Check

$\overrightarrow{AB} \times \overrightarrow{AC} = (18,12,6)$, hence $\vec{n} = (3,2,1)$ and $3x + 2y + z = 6$. Distance $6/\sqrt{14}$; unit normal $(3,2,1)/\sqrt{14}$. At D, $3 + 2 + 1 = 6$, so $D \in \pi$.

The Idea: Dropping a Perpendicular onto a Plane

For any point Q, its **orthogonal projection** H onto π is the point of the plane reached by moving from Q parallel to the normal. If $Q \in \pi$, then $H = Q$.

Visualization



Schematic perspective drawing. Why is this also the *closest* point? For any $R \in \pi$, $QR^2 = QH^2 + HR^2 \geq QH^2$.

Deriving the Projection Formula

Let π pass through P_0 with normal $\vec{n}$, and let Q be any point. Write $\vec{w} = \overrightarrow{P_0Q} = Q - P_0$.

Setting up the condition

The foot H differs from Q by a multiple of the normal: $H = Q - t\vec{n}$ for some scalar t, and H must satisfy the plane equation $\vec{n} \cdot \overrightarrow{P_0H} = 0$.

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Since $\overrightarrow{P_0H} = H - P_0 = (Q - P_0) - t\vec{n} = \vec{w} - t\vec{n}$, the condition becomes

$$\vec{n} \cdot (\vec{w} - t\vec{n}) = 0 \implies \vec{n} \cdot \vec{w} - t(\vec{n} \cdot \vec{n}) = 0 \implies t = \frac{\vec{n} \cdot \vec{w}}{\vec{n} \cdot \vec{n}} = \frac{\vec{n} \cdot \vec{w}}{\|\vec{n}\|^2}$$

The Projection Formula

Orthogonal projection of Q onto π

Let π pass through P_0 with normal $\vec{n}$, and let Q be a point. Set $\vec{w} = Q - P_0$. The orthogonal projection of Q onto π is

$$\boxed{H = Q - t\,\vec{n}, \qquad t = \frac{\vec{n} \cdot \vec{w}}{\vec{n} \cdot \vec{n}}}$$

Reading the formula

Compare with projecting onto a *line*: there we moved **along** the direction vector from P_0; here we move **along the normal** *starting from* Q, since it's the normal — not a vector in the plane — that is perpendicular to π.

Example: Projecting a Point onto a Plane

Project $Q(4,5,6)$ onto the plane $\pi : x + y + z - 3 = 0$ (so $P_0 = (1,1,1) \in \pi$, $\vec{n} = (1,1,1)$).

Example: Projecting a Point onto a Plane

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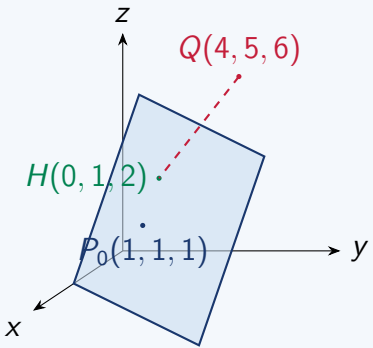
1. $\vec{w} = Q - P_0 = (3,4,5)$.
2. $\vec{n} \cdot \vec{w} = 3 + 4 + 5 = 12, \quad \vec{n} \cdot \vec{n} = 1 + 1 + 1 = 3.$
3. $t = \dfrac{12}{3} = 4.$
4. $H = Q - t\vec{n} = (4,5,6) - 4(1,1,1) = (0,1,2).$

Verify

H on π: $0+1+2-3 = 0$ ✓. $\overrightarrow{HQ} = (4,4,4) \parallel \vec{n}$ ✓: the displacement is purely along the normal.

Projection Picture for the Example

Visualization



$\overrightarrow{HQ} = (4, 4, 4)$ is parallel to the normal $(1, 1, 1)$. Both P_0 and H satisfy $x + y + z = 3$.

Exercise: Projection onto a Plane

Project $Q(1, 2, -3)$ onto $\pi : 2x - y + 2z + 3 = 0$. Before calculating, is Q on the positive or negative side of the plane relative to $\vec{n} = (2, -1, 2)$?

Exercise: Projection onto a Plane

Project $Q(1, 2, -3)$ onto $\pi : 2x - y + 2z + 3 = 0$. Before calculating, is Q on the positive or negative side of the plane relative to $\vec{n} = (2, -1, 2)$?

Check

The equation evaluated at Q gives $2 - 2 - 6 + 3 = -3 < 0$. With $\|\vec{n}\|^2 = 9$, $t = -3/9 = -1/3$ and

$$H = Q - t\vec{n} = (1, 2, -3) + \tfrac{1}{3}(2, -1, 2) = (\tfrac{5}{3}, \tfrac{5}{3}, -\tfrac{7}{3}).$$

Verify: $2(5/3) - 5/3 + 2(-7/3) + 3 = 0$, and $H - Q = \vec{n}/3$.

From Projection to Distance

Big idea

The distance from Q to the plane π is simply $\|H - Q\|$ where H is the foot of the perpendicular. Since $H - Q = -t\vec{n}$ with $t = \dfrac{\vec{n} \cdot \vec{w}}{\vec{n} \cdot \vec{n}}$, we get a clean shortcut without ever computing H:

$$d(Q, \pi) = \|-t\vec{n}\| = |t| \, \|\vec{n}\| = \frac{|\vec{n} \cdot \vec{w}|}{\|\vec{n}\|^2} \, \|\vec{n}\| = \frac{|\vec{n} \cdot \vec{w}|}{\|\vec{n}\|}$$

The Distance Formula

Distance from a point to a plane

For the plane $Ax + By + Cz + D = 0$ and a point $Q(x_0, y_0, z_0)$:

$$\boxed{d(Q, \pi) = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}}$$

Where it comes from

Writing $\vec{n} = (A, B, C)$ and P_0 any point on π, expanding $\vec{n} \cdot \vec{w} = \vec{n} \cdot (Q - P_0) = Ax_0 + By_0 + Cz_0 - (Ax_{P_0} + By_{P_0} + Cz_{P_0}) = Ax_0 + By_0 + Cz_0 + D$, since $P_0 \in \pi$ forces $Ax_{P_0} + By_{P_0} + Cz_{P_0} = -D$.

Example: Distance from a Point to a Plane

Distance from $Q(4, 5, 6)$ to $\pi : x + y + z - 3 = 0$.

Example: Distance from a Point to a Plane

Distance from $Q(4,5,6)$ to $\pi : x + y + z - 3 = 0$.

$$d(Q,\pi) = \frac{|1(4) + 1(5) + 1(6) - 3|}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{|12|}{\sqrt{3}} = \frac{12}{\sqrt{3}} = 4\sqrt{3}$$

Cross-check with the projection

From the earlier projection, $H = (0,1,2)$, and indeed $\|Q - H\| = \|(4,4,4)\| = \sqrt{48} = 4\sqrt{3}$.
✓ Same answer, two routes.

Special Case: Distance from the Origin

A familiar formula

Setting $Q = O = (0, 0, 0)$ recovers exactly the Hesse-form distance from before:

$$d(O, \pi) = \frac{|D|}{\sqrt{A^2 + B^2 + C^2}} = p$$

Example

$\pi : 2x - 2y + z - 6 = 0$: $d(O, \pi) = \frac{|-6|}{3} = 2$, matching $p = 2$ computed earlier.

Exercise: Distance to a Plane

Exercise

1. Find the distance from $Q(2, -1, 3)$ to the plane $3x - 4z + 12 = 0$.
2. A plane has normal $(0, 3, 4)$ and passes through $(1, 0, 0)$. Find the distance from $(1, 5, 5)$ to this plane.

Exercise: Distance to a Plane

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1. Find the distance from $Q(2,-1,3)$ to the plane $3x-4z+12=0$.
2. A plane has normal $(0,3,4)$ and passes through $(1,0,0)$. Find the distance from $(1,5,5)$ to this plane.

Answer key

(1) $A=3, B=0, C=-4, D=12$: $d=\dfrac{|3(2)+0-4(3)+12|}{\sqrt{9+16}}=\dfrac{|6-12+12|}{5}=\dfrac{6}{5}$.

(2) Plane: $3(y-0)+4(z-0)=0 \Rightarrow 3y+4z=0$; $d=\dfrac{|3(5)+4(5)|}{5}=\dfrac{35}{5}=7$.

A Line and a Plane: Three Possible Relationships

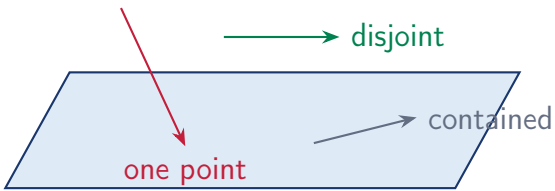
Big idea

A line $L : P_1 + t\vec{d}$ and a plane π with normal $\vec{n}$ can relate in exactly three ways:

- **Transversal:** $\vec{n} \cdot \vec{d} \neq 0$ — the line pierces the plane at **exactly one point**.
- **Parallel, disjoint:** $\vec{n} \cdot \vec{d} = 0$ and $L \not\subset \pi$ — **no intersection**.
- **Contained:** $\vec{n} \cdot \vec{d} = 0$ and $L \subset \pi$ — **every point** of L is an intersection point.

The dot product $\vec{n} \cdot \vec{d}$ measures how much the line's direction points *out of* the plane; it is zero exactly when $\vec{d}$ lies in a direction parallel to π.

Visualizing the Three Cases



In a sketch, the lines may appear to cross in the image even when they are disjoint in space. Use $\vec{n} \cdot \vec{d}$ and the starting point to classify them.

The Method: Substitute and Solve

Finding the intersection point

Let $L : (x, y, z) = P_1 + t\vec{d}$ and $\pi : Ax + By + Cz + D = 0$.

1. Substitute the parametric coordinates of L into the equation of π.
2. Solve the resulting **single linear equation in** t.
3. **Unique** t**?** One intersection point — plug it back into L.
   Equation becomes $0 = 0$**?** $L \subset \pi$: infinitely many intersection points.
   Equation becomes a false statement (like $0 = 5$**)?** No solution: $L \parallel \pi$, disjoint.

Example: Transversal Case (Unique Intersection)

Line $L : (x, y, z) = (1, 0, -1) + t(2, 1, 3)$; plane $\pi : x + y + z - 3 = 0$.

Example: Transversal Case (Unique Intersection)

Line $L : (x, y, z) = (1, 0, -1) + t(2, 1, 3)$; plane $\pi : x + y + z - 3 = 0$.

$$(1 + 2t) + (t) + (-1 + 3t) - 3 = 0$$
$$6t - 3 = 0 \implies t = \tfrac{1}{2}$$

Conclusion

Unique solution $t = \frac{1}{2}$, so L meets π at exactly one point:

$$(1, 0, -1) + \tfrac{1}{2}(2, 1, 3) = \left(2, \tfrac{1}{2}, \tfrac{1}{2}\right)$$

Check: $2 + \frac{1}{2} + \frac{1}{2} - 3 = 0$ ✓.

Example: Parallel and Disjoint (No Intersection)

Line $L : (x, y, z) = (0, 0, 0) + t(1, -1, 0)$; plane $\pi : x + y + z - 3 = 0$.

Example: Parallel and Disjoint (No Intersection)

Line $L : (x, y, z) = (0, 0, 0) + t(1, -1, 0)$; plane $\pi : x + y + z - 3 = 0$.

$$(t) + (-t) + (0) - 3 = 0 \implies -3 = 0 \quad \text{(false, for every } t\text{)}$$

Conclusion

No value of t works: L never meets π. Confirm with normals: $\vec{n} \cdot \vec{d} = (1)(1) + (1)(-1) + (1)(0) = 0$, so L is parallel to π; and $(0, 0, 0)$ does not satisfy $x + y + z - 3 = 0$, so $L \not\subset \pi$. **Parallel, disjoint.**

Example: The Line Lies Inside the Plane

Line $L : (x, y, z) = (1, 1, 1) + t(1, -1, 0)$; plane $\pi : x + y + z - 3 = 0$.

Example: The Line Lies Inside the Plane

Line $L : (x, y, z) = (1, 1, 1) + t(1, -1, 0)$; plane $\pi : x + y + z - 3 = 0$.

$$(1 + t) + (1 - t) + 1 - 3 = 0 \implies 0 = 0 \quad \text{(true for every } t\text{)}$$

Conclusion

The equation holds identically: **every** point of L lies on π, so $L \subset \pi$. As with the parallel case, $\vec{n} \cdot \vec{d} = 1 - 1 + 0 = 0$; the difference is that the starting point $(1, 1, 1)$ *does* satisfy the plane's equation.

Exercise: Line Meets Plane

For each pair, determine whether the line meets the plane at one point, is parallel and disjoint, or lies in the plane.

1. $L : (x, y, z) = (2, 1, 0) + t(1, 0, 1), \quad \pi : 2x - z - 4 = 0.$
2. $L : (x, y, z) = (0, 1, 0) + t(3, 0, -1), \quad \pi : x + 3z - 0 = 0.$

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2. $L : (x, y, z) = (0, 1, 0) + t(3, 0, -1), \quad \pi : x + 3z - 0 = 0.$

Answer key

(1) $2(2 + t) - (t) - 4 = t \Rightarrow t = 0$: one point, $(2, 1, 0)$. (2) $(3t) + 3(-t) = 0$ for all t; and $(0, 1, 0)$ satisfies $x + 3z = 0$, so $L \subset \pi$.

Two Planes: Three Possible Relationships

Big idea

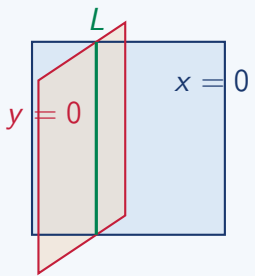
Two planes π_1 (normal $\vec{n}_1$) and π_2 (normal $\vec{n}_2$) can relate in exactly three ways:

- **Equal:** same plane, infinitely many shared points (a whole 2D set).
- **Parallel, distinct:** $\vec{n}_1 \parallel \vec{n}_2$ but different planes — **no intersection**.
- **Transversal:** $\vec{n}_1 \nparallel \vec{n}_2$ — they meet in a **whole line**.

Unlike two lines in $\mathbb{R}^3$, two planes can never be skew: if their normals aren't parallel, they *must* cross, in a line.

Visualizing Two Intersecting Planes

Visualization



In this example $x = 0$ and $y = 0$ meet along $L : (x, y, z) = (0, 0, t)$. Their normals $(1, 0, 0)$ and $(0, 1, 0)$ are not parallel.

Step 1: Test the Normals

Classifying the relationship

1. Compute $\vec{n}_1 \times \vec{n}_2$.
2. $\vec{n}_1 \times \vec{n}_2 = \vec{0}$? The normals are parallel. Check whether a point of π_1 satisfies π_2's equation: **yes** $\Rightarrow$ equal planes; **no** $\Rightarrow$ parallel and distinct.
3. $\vec{n}_1 \times \vec{n}_2 \neq \vec{0}$? The planes are transversal; their intersection is a line with **direction vector** $\vec{d} = \vec{n}_1 \times \vec{n}_2$ (perpendicular to both normals, hence lying in both planes).

Example: Parallel, Distinct Planes

$\pi_1 : x + 2y - z - 1 = 0, \quad \pi_2 : 2x + 4y - 2z - 10 = 0.$

Example: Parallel, Distinct Planes

$$\pi_1 : x + 2y - z - 1 = 0, \qquad \pi_2 : 2x + 4y - 2z - 10 = 0.$$

$$\vec{n}_1 = (1, 2, -1), \qquad \vec{n}_2 = (2, 4, -2) = 2\vec{n}_1 \quad \Rightarrow \quad \vec{n}_1 \times \vec{n}_2 = \vec{0}.$$

Conclusion

The normals are parallel. Take $P_0(1, 0, 0) \in \pi_1$ (since $1 + 0 - 0 - 1 = 0$) and test in π_2: $2(1) + 4(0) - 2(0) - 10 = -8 \neq 0$. So $P_0 \notin \pi_2$: the planes are **parallel and distinct** — no intersection.

Equal Planes: Check the Constant Too

Classify before solving

Compare $\pi_1 : x + 2y - z - 1 = 0$ with $\pi_2 : 2x + 4y - 2z - 2 = 0$. How does this pair differ from the earlier disjoint pair with constant -10?

Equal Planes: Check the Constant Too

Classify before solving

Compare $\pi_1 : x + 2y - z - 1 = 0$ with $\pi_2 : 2x + 4y - 2z - 2 = 0$. How does this pair differ from the earlier disjoint pair with constant -10?

Check

π_2 is exactly twice π_1, *including the constant*. Hence the planes are equal: every point of either plane is common to both. Parallel normals alone cannot distinguish equality from disjointness.

Example: Finding the Line of Intersection

$\pi_1 : x + y + z - 3 = 0, \qquad \pi_2 : x - y + 2z - 2 = 0.$

Predict

Before computing: do the normals look proportional? What shape is the intersection?

Example: Finding the Line of Intersection

$\pi_1 : x + y + z - 3 = 0, \qquad \pi_2 : x - y + 2z - 2 = 0.$

Predict

Before computing: do the normals look proportional? What shape is the intersection?

$$\vec{n}_1 = (1, 1, 1), \quad \vec{n}_2 = (1, -1, 2).$$

$$\boxed{\vec{d} = \vec{n}_1 \times \vec{n}_2 = (3, -1, -2) \neq \vec{0}.}$$

Check the direction: $\vec{n}_1 \cdot \vec{d} = 3 - 1 - 2 = 0$ and $\vec{n}_2 \cdot \vec{d} = 3 + 1 - 4 = 0$. We still need one point on both planes.

Finding a Point on the Line of Intersection

Fix a coordinate, then solve

Choose a convenient value for one coordinate, and solve the remaining 2×2 system. If that value gives no point, try another coordinate or a different value.

Continuing the example

Set $z = 0$: $x + y = 3$ and $x - y = 2$. Hence $x = \frac{5}{2}$, $y = \frac{1}{2}$ and $P_0 = (\frac{5}{2}, \frac{1}{2}, 0)$ satisfies both equations.

Why can a trial fail? The line $x = 0, z = 1$ has no point with $z = 0$, although its defining planes do intersect.

The Line of Intersection, Assembled

Point + direction

$$\boxed{L : (x, y, z) = \left(\tfrac{5}{2}, \tfrac{1}{2}, 0\right) + t(3, -1, -2), \qquad t \in \mathbb{R}.}$$

Verify a whole line, not just one point

Substitution into π_1 gives $(5/2+3t)+(1/2-t)-2t-3=0$. Into π_2 it gives $(5/2+3t)-(1/2-t)+2(-2t)-2=0$. Both hold for *every* t.

Exercise: Intersection of Two Planes

$\pi_1 : 2x - y + z - 4 = 0, \quad \pi_2 : x + y - 2z + 1 = 0.$

1. Show the planes are not parallel.
2. Find a direction vector and a point on the line of intersection.

Exercise: Intersection of Two Planes

$\pi_1 : 2x - y + z - 4 = 0, \quad \pi_2 : x + y - 2z + 1 = 0.$

1. Show the planes are not parallel.
2. Find a direction vector and a point on the line of intersection.

Answer key

$\vec{n}_1 = (2, -1, 1)$, $\vec{n}_2 = (1, 1, -2)$: $\vec{d} = \vec{n}_1 \times \vec{n}_2 = \big((-1)(-2) - 1(1),\ 1(1) - 2(-2),\ 2(1) - (-1)(1)\big) = (1, 5, 3) \neq \vec{0}$, so the planes are not parallel. Setting $z = 0$: $2x - y = 4$, $x + y = -1$; adding gives $3x = 3 \Rightarrow x = 1$, $y = -2$. Point $(1, -2, 0)$.

Review Exercises

1. Through $A(1,2,0)$, $B(3,0,1)$, $C(0,1,4)$: find a Cartesian equation and the distance from O.
2. Where does $(x,y,z) = (0,0,5) + t(1,1,-1)$ meet $x+y+z=1$?
3. Find the line common to $x-2y+z=0$ and $2x+y-z=3$.
4. Project O onto $x-y+2z-6=0$; check the distance in two ways.

For each answer, show at least one substitution or dot-product check.

Answers

1. $\overrightarrow{AB} = (2, -2, 1)$, $\overrightarrow{AC} = (-1, -1, 4)$; cross product $(-7, -9, -4)$, so $7x + 9y + 4z - 25 = 0$ and $d(O, \pi) = 25/\sqrt{146}$.
2. $5 + t = 1 \Rightarrow t = -4$; intersection $(-4, -4, 9)$.
3. $\vec{n}_1 \times \vec{n}_2 = (1, 3, 5)$; setting $z = 0$ gives $(x, y) = (6/5, 3/5)$, hence $(x, y, z) = (6/5, 3/5, 0) + t(1, 3, 5)$.
4. $\vec{n} = (1, -1, 2)$, $\|\vec{n}\|^2 = 6$; $H = O - (-6/6)\vec{n} = (1, -1, 2)$, distance $\sqrt{6} = 6/\sqrt{6}$.